Weekly Assignment 4

WRITE YOUR NAME HERE
MATH 117: Advanced Linear Algebra

August 26, 2023

In this assignment, you will prove the Cayley-Hamilton theorem. Throughout, V is a finite-dimensional vector space and $L : V \to V$ is a linear operator. See the weekly assignment webpage for due dates, templates, and assignment description. Make sure to justify any claims you make. You may not appeal to any results that we have not discussed in class.

1. Suppose that W is an L-invariant[1] subspace of V. Then we obtain by restriction a linear operator $L|_W : W \to W$. Prove that the characteristic polynomial of $L|_W$ divides the characteristic polynomial of L.[2]

2. For $v \neq 0$, define $\operatorname{span}(L, v) := \operatorname{span}\{L^i(v) : i \in \mathbb{N}_0\}$. Here $L^0 := \mathrm{Id}_V$.

(a) Prove that $\operatorname{span}(L, v)$ is an L-invariant subspace of V.

(b) Prove that there exists a largest integer k such that $B_k := \{v, L(v), \ldots, L^{k-1}(v)\}$ is independent. Moreover, show that B_k is a basis for $\operatorname{span}(L, v)$.

3. Let $v \neq 0$ and let $B := B_k$ be the basis for $W := \operatorname{span}(L, v)$ from Problems 1 & 2. Define $m_{L,v}(x) = a_0 + a_1 x + \cdots + a_{k-1}x^{k-1} + x^k \in F[x]$ where $a_0, a_1, \ldots, a_{k-1}$ are the unique coefficients such that

$$a_0 v + a_1 L(v) + \cdots + a_{k-1} L^{k-1}(v) + L^k(v) = 0.$$

Since W is L-invariant, we obtain by restriction a linear operator $L|_W : W \to W$. The goal of this problem is to show that the characteristic polynomial of $L|_W$ is equal to $m_{L,v}(x)$.

(a) Compute the matrix $xI_k - [L|_W]_B$.

(b) Show that $\det(xI_k - [L|_W]_B) = m_{L,v}(x)$ using induction on k. [3]

Proof. Write your proof here. □

4. Use Problems 1-3 to prove the Cayley-Hamilton theorem: the linear operator L is a root of its characteristic polynomial, that is, $c_L(L)$ is the zero operator in $\operatorname{End}(V)$.[4]

Proof. Write your proof here. □

[1] See Def 4.3.6

[2] Hint: start with a basis for W and extend to a basis B for V. Argue that $[L]_B = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$ for some matrices X, Y, Z. Compute $c_L(x)$ using this fact.

[3] Hint: For the inductive step, start by using cofactor expansion along the first row.

[4] Hint: need to show $c_L(L)(v) = 0$ for all $v \in V$. Case 1: $v = 0$. Case 2: $v \neq 0$, invoke Problems 1 & 3 to write $c_L(x)$ as a product of two polynomials. Use this to show $c_L(L)(v) = 0$.