Weekly Assignment 3

WRITE YOUR NAME HERE

MATH 117: Advanced Linear Algebra

August 16, 2023

Some hints for this assignment are written in the footnotes. See the weekly assignment webpage for due dates, templates, and assignment description. Make sure to justify any claims you make. You may not appeal to any results that we have not discussed in class.

1. Let V be finite-dimensional vector space and W a subspace. Suppose that $\{b_1, \ldots, b_k\}$ is a basis for W and extend this to a basis $\{b_1, \ldots, b_k, b_{k+1}, \ldots, b_n\}$ for V using Proposition 1.4.11. Prove that the set of vectors $\{b_{k+1} + W, \ldots, b_n + W\}$ is a basis for the quotient space V/W.

Proof. Write your proof here. □

Definition 1. *Let V be a vector space and let S be a subset of V. The* annihilator *S^0 of S is the set of linear functionals whose kernel contains S, that is,*

$$S^0 := \{f \in V^* : f(v) = 0 \text{ for all } v \in S\} \subset V^*.$$

2. Let V be a vector space.

(a) Let S be a subset of V. Prove that S^0 is a subspace of V^*.

(b) Let W be a subspace of V. Prove that W^0 is isomorphic to $(V/W)^*$.[1]

(c) Suppose that V is finite-dimensional and let W be a subspace of V. By part (b), $\dim(W^0) = \dim V - \dim W$. Provide another proof of this equation using dual bases.[2]

Proof. Write your proof here. □

Definition 2. *Let V be a vector space. A bilinear form $B : V \times V \to F$ is called* reflexive *if $B(v, v') = 0$ implies $B(v', v) = 0$ for all $v, v' \in V$. The* radical *of a reflexive bilinear form is the set*

$$\mathrm{rad}(V) := \{v \in V : B(v, v') = 0 \text{ for all } v' \in V\}.$$

A reflexive bilinear form is called nondegenerate *if $\mathrm{rad}(V) = \{0\}$.*

3. Let V be a vector space. Let $B : V \times V \to F$ be a bilinear form on V.

[1] Hint: Universal Property of the Quotient.

[2] Hint: Start with a basis for W and extend to a basis for V. Can you use the corresponding dual basis to construct a basis for W^0?

(a) For any $v \in V$, define a function $\Phi_B(v) : V \to F$ by the rule $(\Phi_B(v))(w) = B(v, w)$. Show that $\Phi_B(v)$ is a linear functional and show that the assignment $v \mapsto \Phi_B(v)$ defines a linear map $\Phi_B : V \to V^*$.

(b) Suppose that V is finite-dimensional and that B is reflexive and nondegenerate.

(i) Prove that Φ_B is an isomorphism.

(ii) Let W be a subspace of V. Describe the preimage $W^\perp := \Phi_B^{-1}(W^0)$ of W^0 under Φ_B. In particular, $W^0 \cong W^\perp$.

(iii) Suppose that B is nondegenerate when restricted to W, i.e., $\operatorname{rad}(W) = \{0\}$. Prove that $V = W \oplus W^\perp$.

Proof. Write your proof here. $\square$

4. (i) Suppose that $L_1 : V_1 \to W_1$ and $L_2 : V_2 \to W_2$ are linear maps. Prove that there is a unique linear map

$$L_1 \otimes L_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$$

with the property that $(L_1 \otimes L_2)(v_1 \otimes v_2) = L(v_1) \otimes L(v_2)$ for all $v_1 \in V_1$ and $v_2 \in V_2$.[3]

(ii) Let $F = \mathbb{Z}_5$ and let $V = F^2$. Let $L : V \to V$ be the linear map defined by left multiplication with the matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ 4 & 2 \end{smallmatrix}\right)$. Let $E = (e_1, e_2)$ denote the standard basis for V. Compute the matrix

$$[L \otimes L]_B$$

for linear map $L \otimes L : V \otimes V \to V \otimes V$, where B is the basis $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$ for $V \otimes V$.

Proof. Write your proof here. $\square$

[3]Hint: Universal Property of the Tensor Product.