Weekly Assignment 2 Solutions

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MATH 117: Advanced Linear Algebra

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Some hints for this assignment are written in the footnotes. See the weekly assignment webpage for due dates, templates, and assignment description. Make sure to justify any claims you make. You may not appeal to any results that we have not discussed in class.

1. Prove Proposition 2.1.7: Let V and W be vector spaces over F and let $L : V \to W$ be a linear map. Then

(a) $\mathrm{im}(L)$ is a subspace of W; and

(b) $\ker(L)$ is a subspace of V.

Proof. We need to show that each set is nonempty, closed under addition, and closed under scalar multiplication.

(a) We have $0_W \in \mathrm{im}(L)$ since $L(0_V) = 0_W$. Let $w, z \in \mathrm{im}(L)$ and let $\alpha \in F$. Choose $u, v \in V$ such that $L(u) = w$ and $L(v) = z$. Then

$$L(\alpha u + v) = \alpha L(u) + L(v) = \alpha w + z.$$

This proves that $\alpha w + z \in \mathrm{im}(L)$. This proves simultaneously that $\mathrm{im}(L)$ is closed under addition (let $\alpha = 1$) and scalar multiplication (let $v = 0_V$).

(b) We have $0_V \in \ker(L)$ since $L(0_V) = 0_W$. Let $u, v \in \ker(L)$ and let $\alpha \in F$. Then

$$L(\alpha u + v) = \alpha L(u) + L(v) = \alpha \cdot 0_W + 0_W = 0_W.$$

Thus, $\alpha u + v \in \ker(L)$.

This proves that $\mathrm{im}(L)$ and $\ker(L)$ are subspaces of W and V, respectively. □

2. Prove Theorem 2.5.1: Suppose that V, W are vector spaces over F and $L, K : V \to W$ are linear maps. Then $\alpha L + \beta K$ is a linear map for all $\alpha, \beta \in F$.[1]

Proof. Let $u, v \in V$ and $\gamma \in F$. Then

$$\begin{aligned}
(\alpha L + \beta K)(u + v) &= (\alpha L)(u + v) + (\beta K)(u + v) && \text{(Def. of addition in Hom}(V, W)\text{)} \\
&= \alpha L(u + v) + \beta K(u + v) && \text{(Def. of scalar mult. in Hom}(V, W)\text{)} \\
&= \alpha(L(u) + L(v)) + \beta(K(u) + K(v)) && (L, K \text{ are linear maps}) \\
&= \alpha L(u) + \beta K(u) + \alpha L(v) + \beta K(v) && \text{(vector space properties)} \\
&= (\alpha L + \beta K)(u) + (\alpha L + \beta K)(v). && \text{(Def. of Hom}(V, W)\text{ again)}
\end{aligned}$$

[1] Since the zero map is linear, this implies that $\mathrm{Hom}(V, W)$ is a subspace of W^V.

I tried to make it as clear as possible what I am doing at each step. Your proof might not explain each step or might combine multiple steps into one - this is fine, as long as its clear to the reader what is happening. The computation for $(\alpha L + \beta K)(\gamma u)$ is similar, so I omit it. You need to use associativity of scalar multiplication in W and also commutativity of multiplication in F.

$\square$

3. Let V_1, V_2 be vector spaces over F. The direct sum $V_1 \oplus V_2$ comes with linear maps

$$\iota_1 : V_1 \to V_1 \oplus V_2,\ v_1 \mapsto (v_1, 0) \quad \text{and} \quad \iota_2 : V_2 \to V_1 \oplus V_2,\ v_2 \mapsto (0, v_2).$$

Let Z be any other vector space and let $L_1 : V_1 \to Z$ and $L_2 : V_2 \to Z$ be any other linear maps. Prove that there is a unique linear map $L : V_1 \oplus V_2 \to Z$ with the property that $L \circ \iota_1 = L_1$ and $L \circ \iota_2 = L_2$. See Remark 1.

Proof. Define $L : V_1 \oplus V_2 \to Z$ via the rule $L(v_1, v_2) = L_1(v_1) + L_2(v_2)$. It is easy to verify that L is linear using the fact that both L_1 and L_2 are linear. Let $v_1 \in V_1$. Then

$$(L \circ \iota_1)(v_1) = L((v_1, 0)) = L_1(v_1) + L_2(0) = L_1(v_1)$$

which shows that $L \circ \iota_1 = L_1$. By a symmetric argument, we can also conclude that $L \circ \iota_2 = L_2$. Thus, $L : V_1 \oplus V_2 \to Z$ is a linear map with the desired property. This proves existence. Suppose $K : V_1 \oplus V_2 \to Z$ is another linear map satisfying the same property. Then for any $(v_1, v_2) \in V_1 \oplus V_2$, we have

$$K(v_1, v_2) = K(\iota_1(v_1) + \iota_2(v_2)) = (K \circ \iota_1)(v_1) + (K \circ \iota_2)(v_2) = L_1(v_1) + L_2(v_2) = L(v_1, v_2).$$

Thus, $K = L$ which proves that the map L is unique. $\square$

4. Let V_1, V_2, W_1, W_2 be vector spaces over F and let $L_1 : V_1 \to W_1$ and $L_2 : V_2 \to W_2$ be linear maps.

(a) Use the Universal Property of the Direct Sum (see Remark 1) to show that there is a unique linear map

$$L_1 \oplus L_2 : V_1 \oplus V_2 \to W_1 \oplus W_2$$

satisfying $(L_1 \oplus L_2)(v_1, v_2) = (L_1(v_1), L_2(v_2))$.

(b) Suppose additionally that V_1, V_2, W_1, W_2 are finite-dimensional with ordered bases $B_1 = (a_1, \ldots, a_k)$, $B_2 = (b_1, \ldots, b_l)$, $C_1 = (c_1, \ldots, c_m)$, and $C_2 = (d_1, \ldots, d_n)$, respectively.

  (i) Prove that $B := ((a_1, 0), \ldots, (a_k, 0), (0, b_1), \ldots, (0, b_l))$ is a basis for $V_1 \oplus V_2$. Similarly, $C := ((c_1, 0), \ldots, (c_l, 0), (0, d_1), \ldots, (0, d_n))$ is a basis for $W_1 \oplus W_2$.

  (ii) Prove that the matrix for $L_1 \oplus L_2$ with respect to B and C has the following block diagonal form:

$$[L_1 \oplus L_2]_B^C = \begin{pmatrix} [L_1]_{B_1}^{C_1} & 0 \\ 0 & [L_2]_{B_2}^{C_2} \end{pmatrix}.$$

Proof. (a) Denote the inclusions maps for each direct sum by

$$\iota_{V_i} : V_i \to V_1 \oplus V_2 \text{ and } \iota_{W_i} : W_i \to W_1 \oplus W_2$$

where $i \in \{1, 2\}$. Then we have linear maps $\iota_{W_i} \circ L_i : V_i \to W_1 \oplus W_2$ for $i \in \{1, 2\}$. According the the Universal Property of the Direct Sum, there is a unique linear map

$$L_1 \oplus L_2 : V_1 \oplus V_2 \to W_1 \oplus W_2$$

satisfying

$$(L_1 \oplus L_2) \circ \iota_{V_1} = \iota_{W_1} \circ L_1 \text{ and } (L_1 \oplus L_2) \circ \iota_{V_2} = \iota_{W_2} \circ L_2.$$

We just need to show that $(L_1 \oplus L_2)(v_1, v_2) = (L_1(v_1), L_2(v_2))$. For any $(v_1, v_2) \in V_1 \oplus V_2$, we have

$$\begin{aligned}
(L_1 \oplus L_2)(v_1, v_2) &= (L_1 \oplus L_2)(\iota_{V_1}(v_1) + \iota_{V_2}(v_2)) \\
&= (L_1 \oplus L_2)(\iota_{V_1}(v_1)) + (L_1 \oplus L_2)(\iota_{V_2}(v_2)) \\
&= (\iota_{W_1} \circ L_1)(v_1) + (\iota_{W_1} \circ L_1)(v_2) \\
&= (L_1(v_1), L_2(v_2)).
\end{aligned}$$

This proves the claim.

(b) (i) To prove (i), it suffices to show that B is spanning (or equivalently independent) because $|B| = k + l = \dim V_1 + \dim V_2 = \dim V_1 \oplus V_2$. Either of these claims are straightforward to show.

(ii) To prove (ii), you could work directly from the definitions to show that the matrices have the same columns. A more elegant solution would be to use the uniqueness property of the matrix $[L_1 \oplus L_2]_B^C$ - it is the unique matrix satisfying

$$[L_1 \oplus L_2]_B^C [(v_1, v_2)]_B = [(L_1 \oplus L_2)(v_1, v_2)]_C$$

for all $v_1 \in V_1$ and $v_2 \in V_2$. Let $v_1 \in V_1$ and $v_2 \in V_2$. Then $[(v_1, v_2)]_B = \begin{pmatrix} [v_1]_{B_1} \\ [v_2]_{B_2} \end{pmatrix}$. Hence,

$$\begin{aligned}
\begin{pmatrix} [L_1]_{B_1}^{C_1} & 0 \\ 0 & [L_2]_{B_2}^{C_2} \end{pmatrix} [(v_1, v_2)]_B &= \begin{pmatrix} [L_1]_{B_1}^{C_1} & 0 \\ 0 & [L_2]_{B_2}^{C_2} \end{pmatrix} \begin{pmatrix} [v_1]_{B_1} \\ [v_2]_{B_2} \end{pmatrix} \\
&= \begin{pmatrix} [L_1]_{B_1}^{C_1} [v_1]_{B_1} \\ [L_2]_{B_2}^{C_2} [v_2]_{B_2} \end{pmatrix} \\
&= \begin{pmatrix} [L_1(v_1)]_{C_1} \\ [L_2(v_2)]_{C_2} \end{pmatrix} \\
&= [(L_1(v_1), L_2(v_2))]_C \\
&= [(L_1 \oplus L_2)(v_1, v_2)]_C.
\end{aligned}$$

By uniqueness, we conclude that $[L_1 \oplus L_2]_B^C = \begin{pmatrix} [L_1]_{B_1}^{C_1} & 0 \\ 0 & [L_2]_{B_2}^{C_2} \end{pmatrix}$.

$\square$

Remark 1. *In other words, L is the unique linear map making the following diagram commute*

$$\begin{tikzcd}
V_1 \arrow[dr, "\iota_1"'] \arrow[drr, bend left, "L_1"] & & \\
& V_1 \oplus V_2 \arrow[r, dashed, "\exists! L"] & Z \\
V_2 \arrow[ur, "\iota_2"] \arrow[urr, bend right, "L_1"'] & &
\end{tikzcd}$$

The property described in Problem 1 is usually called the Universal Property of the Direct Product. It provides a precise answer to the question "How do I define a linear map out of the direct sum of two vector spaces?". One simply defines linear maps L_1 and L_2 as in the statement. Your proof will *provide the recipe for constructing the desired linear map L. Moreover, it establishes a bijection of sets*

$$\mathrm{Hom}_F(V_1 \oplus V_2, Z) \longleftrightarrow \mathrm{Hom}_F(V_1, Z) \oplus \mathrm{Hom}_F(V_2, Z)$$

$$L \longmapsto (L \circ \iota_1, L \circ \iota_2)$$

However, notice that the domain and codomain are actually vector spaces. It can be shown that this bijection is a linear map, i.e., a vector space isomorphism. Note that all of this readily generalizes to the direct sum of finitely many vector spaces. Compare all of this with the discussion on the Universal Property of the Quotient Space from the lecture.