Weekly Assignment 2

WRITE YOUR NAME HERE
MATH 117: Advanced Linear Algebra

August 9, 2023

Some hints/comments for this assignment may be written in the footnotes. See the weekly assignment webpage for due dates, templates, and assignment description. Make sure to justify any claims you make. You may not appeal to any results that we have not discussed in class.

1. Prove Proposition 2.1.7: Let V and W be vector spaces over F and let $L : V \to W$ be a linear map. Then

(a) $\mathrm{im}(L)$ is a subspace of W; and

(b) $\ker(L)$ is a subspace of V.

Proof. Write your proof here. □

2. Prove Theorem 2.5.1: Suppose that V, W are vector spaces over F and $L, K : V \to W$ are linear maps. Then $\alpha L + \beta K$ is a linear map for all $\alpha, \beta \in F$.[1]

Proof. Write your proof here. □

3. Let V_1, V_2 be vector spaces over F. The direct sum $V_1 \oplus V_2$ comes with linear maps

$$\iota_1 : V_1 \to V_1 \oplus V_2,\ v_1 \mapsto (v_1, 0) \quad \text{and} \quad \iota_2 : V_2 \to V_1 \oplus V_2,\ v_2 \mapsto (0, v_2).$$

Let Z be any other vector space and let $L_1 : V_1 \to Z$ and $L_2 : V_2 \to Z$ be any other linear maps. Prove that there is a unique linear map $L : V_1 \oplus V_2 \to Z$ with the property that $L \circ \iota_1 = L_1$ and $L \circ \iota_2 = L_2$. See Remark 1.

Proof. Write your proof here. □

4. Let V_1, V_2, W_1, W_2 be vector spaces over F and let $L_1 : V_1 \to W_1$ and $L_2 : V_2 \to W_2$ be linear maps.

(a) Use the Universal Property of the Direct Sum (see Remark 1) to show that there is a unique linear map

$$L_1 \oplus L_2 : V_1 \oplus V_2 \to W_1 \oplus W_2$$

satisfying $(L_1 \oplus L_2)(v_1, v_2) = (L_1(v_1), L_2(v_2))$.

[1] Since the zero map is linear, this implies that $\mathrm{Hom}(V, W)$ is a subspace of W^V.

(b) Suppose additionally that V_1, V_2, W_1, W_2 are finite-dimensional with ordered bases $B_1 = (a_1, \ldots, a_k)$, $B_2 = (b_1, \ldots, b_l)$, $C_1 = (c_1, \ldots, c_m)$, and $C_2 = (d_1, \ldots, d_n)$, respectively.

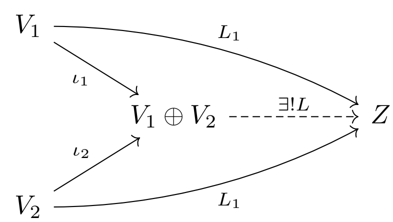
(i) Prove that $B := ((a_1, 0), \ldots, (a_k, 0), (0, b_1), \ldots, (0, b_l))$ is a basis for $V_1 \oplus V_2$. Similarly, $C := ((c_1, 0), \ldots, (c_l, 0), (0, d_1), \ldots, (0, d_n))$ is a basis for $W_1 \oplus W_2$.

(ii) Prove that the matrix for $L_1 \oplus L_2$ with respect to B and C has the following block diagonal form:

$$[L_1 \oplus L_2]_B^C = \begin{pmatrix} [L_1]_{B_1}^{C_1} & 0 \\ 0 & [L_2]_{B_2}^{C_2} \end{pmatrix}.$$

Proof. Write your proof here. □

Remark 1. *In other words, L is the unique linear map making the following diagram commute*



The property described in Problem 1 is usually called the Universal Property of the Direct Product. It provides a precise answer to the question "How do I define a linear map out of the direct sum of two vector spaces?". One simply defines linear maps L_1 and L_2 as in the statement. Your proof will provide the recipe for constructing the desired linear map L. Moreover, it establishes a bijection of sets

$$\mathrm{Hom}_F(V_1 \oplus V_2, Z) \longleftrightarrow \mathrm{Hom}_F(V_1, Z) \oplus \mathrm{Hom}_F(V_2, Z)$$

$$L \longmapsto (L \circ \iota_1, L \circ \iota_2)$$

However, notice that the domain and codomain are actually vector spaces. It can be shown that this bijection is a linear map, i.e., a vector space isomorphism.