Weekly Assignment 1 Solutions

Jadyn V. Breland
MATH 117: Advanced Linear Algebra

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Some hints for this assignment are written in the footnotes. See the weekly assignment webpage for due dates, templates, and assignment description.

1. Let F be a field and let $V = F$. Denote the additive and multiplicative identities of F by 0_F and 1_F, respectively. For $u, v \in V$ and $\alpha \in F$, define vector addition by $u \oplus v := u + v - 1_F$ and scalar multiplication by $\alpha \odot u := \alpha u - \alpha + 1_F$. Prove that $(V, \oplus, \odot)$ is an F-vector space.[1]

Proof. In this proof, I will freely use the fact that F is a field without explicitly mentioning which property I used.

The additive identity with respect to $\oplus$ is given by $0_V := 1_F$. Indeed, let $v \in F$. Then $1_F \oplus v = 1_F + v - 1_F = v = v + 1_F - 1_F = v \oplus 1_F$. The additive inverse of $v \in V$ is given by $\ominus v := 1_F + 1_F - v$[2] because

$$\begin{aligned} v \oplus (1_F + 1_F - v) &= v + (1_F + 1_F - v) - 1_F \\ &= 1_F \\ &= (1_F + 1_F - v) + v - 1_F \\ &= (1_F + 1_F - v) \oplus v. \end{aligned}$$

The binary operation $\oplus$ is clearly associative and commutative because they are defined in terms of the binary operations on F, which are associative and commutative. Thus, $(V, \oplus)$ is an abelian group.

It remains to check conditions V1-V3 from Definition 1.2.1. Let $v, w \in V$ and $\alpha, \beta \in F$. Then $1_F \odot v = 1_F \cdot v - 1_F + 1_F = 1_F \cdot v = v$, which verifies $V1$. Further,

$$\begin{aligned} (\alpha\beta) \odot v &= (\alpha\beta)v - \alpha\beta + 1_F \\ &= \alpha(\beta v) - \alpha\beta + \alpha - \alpha + 1_F \\ &= \alpha(\beta v - \beta + 1_F) - \alpha + 1_F \\ &= \alpha \odot (\beta v - \beta + 1_F) \\ &= \alpha \odot (\beta \odot v). \end{aligned}$$

[1] You need to specify a zero vector 0_V and the additive inverse $\ominus u$ of $u \in F$, and then verify the several defining conditions of a vector space.

[2] You could just write 2 instead of $1_F + 1_F$, I am just trying to be as clear as possible.

This verifies V2. Finally,

$$\begin{aligned}\alpha \odot (v \oplus w) &= \alpha \odot (v + w - 1_F) \\ &= \alpha(v + w - 1_F) - \alpha + 1_F \\ &= (\alpha v - \alpha + 1_F) + (\alpha w - \alpha + 1_F) - 1_F \\ &= (\alpha v - \alpha + 1_F) \oplus (\alpha w - \alpha + 1_F) \\ &= (\alpha \odot v) \oplus (\alpha \odot w).\end{aligned}$$

The final equality $(\alpha + \beta) \odot v = (\alpha \odot v) \oplus (\beta \odot v)$ in V3 can also be easily verified. □

2. Suppose that $W_1, \ldots, W_n$ are subspaces of a vector space V over a field F. Prove that

$$\sum_{i=1}^{n} W_i = \left\{ \sum_{i=1}^{n} w_i : w_i \in W_i \text{ for all } i = 1, \ldots, n \right\}.$$

Proof. The proof is by induction on n. By definition, $\sum_{i=1}^{n} W_i = \operatorname{span}\left(\bigcup_{i=1}^{n} W_i\right)$. If $n = 1$, this is just equal to W_1, since the span of a subspace is the subspace itself. Clearly, $W_1 = \{w_1 : w_1 \in W\}$, so the claim is true.

We will need the case $n = 2$ as well. If $n = 2$, then we need to prove that $W_1 + W_2 = W$ where $W := \{w_1 + w_2 : w_1 \in W_1,\ w_2 \in W_2\}$. Notice that W is a subspace of V which contains W_1 and W_2. By definition, $W_1 + W_2$ is the smallest subspace of V containing W_1 and W_2. This implies that $W_1 + W_2 \subseteq W$. On the other hand, let $w \in W$. Then $w = w_1 + w_2$ for some $w_1 \in W_1$ and some $w_2 \in W_2$. Hence,

$$w = w_1 + w_2 \in \operatorname{span}(\{w_1, w_2\}) \subseteq \operatorname{span}(W_1 \cup W_2) = W_1 + W_2.$$

Thus $W = W_1 + W_2$. We will also need to use that fact that $\operatorname{span}(S \cup T) = \operatorname{span}(S) + \operatorname{span}(T)$. Now, assume that the equality of sets is true for any collection of n subspaces. Consider a collection $W_1, \ldots, W_n, W_{n+1}$ of $n + 1$ subspaces. Then

$$\begin{aligned}\sum_{i=1}^{n} W_i &= \operatorname{span}\left(\bigcup_{i=1}^{n} W_i \cup W_{n+1}\right) \\ &= \operatorname{span}\left(\bigcup_{i=1}^{n} W_i\right) + \operatorname{span}(W_{n+1}) \\ &= \sum_{i=1}^{n} W_i + W_{n+1} \\ &= \left\{ \sum_{i=1}^{n} w_i : w_i \in W_i \text{ for all } i = 1, \ldots, n \right\} + W_{n+1} && \text{(Inductive hypothesis)} \\ &= \left\{ \sum_{i=1}^{n} w_i + w_{n+1} : w_i \in W_i,\ i = 1, \ldots, n+1 \right\} && (n = 2 \text{ case}) \\ &= \left\{ \sum_{i=1}^{n+1} w_i : w_i \in W_i \text{ for all } i = 1, \ldots, n+1 \right\}.\end{aligned}$$

This completes the proof. □

3. Prove Proposition 1.4.8: a subset B of a vector space V is a basis if and only if B is a minimal[3] spanning set.

Proof. Suppose that $B \subseteq V$ is a basis. Then B is a spanning set. To prove it is minimal, suppose C is another spanning set for V such that $C \subsetneq B$. Let $b \in B \setminus C$. Since C spans V there is a finite collection of vectors $c_1, \ldots, c_n \in B$ and scalars $a_1, \ldots, a_n \in F$ such that

$$b = a_1 c_1 + \cdots + a_n c_n.$$

But then

$$a_1 c_1 + \cdots + a_n c_n + (-1)b = 0$$

is a nontrivial linear combination of vectors in B (since $b \notin C$). This is a contradiction to the fact that B is independent.

Conversely, suppose that B is a minimal spanning set and suppose to the contrary that B is not a basis. Then B is dependent. Using Proposition 1.4.3, choose $v \in B$ such that $v \in \operatorname{span}(B \setminus \{v\})$. Then $V = \operatorname{span}(B) = \operatorname{span}(B \setminus \{v\})$, contradicting the fact that B is a minimal spanning set. □

4. Let M and N be finite-dimensional subspaces of a (not necessarily finite dimensional) vector space V. Prove the following equation:

$$\dim(M) + \dim(N) = \dim(M + N) + \dim(M \cap N).$$

Proof. Let $u_1 \ldots, u_n$ be a basis for $M \cap N$. Since $M \cap N \subseteq M$ and M is finite dimensional, we can extend this to a basis $u_1 \ldots, u_n, v_1, \ldots, v_k$ for M. Similarly, we can extend to a basis $u_1 \ldots, u_n, w_1, \ldots, w_l$ for N. It suffices to show that the vectors

$$u_1 \ldots, u_n, v_1, \ldots, v_k, w_1, \ldots, w_l$$

are a basis for $M + N$, because then

$$\dim(M) + \dim(N) = (n + k) + (n + l) = (n + k + l) + n = \dim(M + N) + \dim(M \cap N).$$

Clearly, they span $M + N$, because every every vector in $M + N$ is a sum of a vector from M and a vector from N. So it suffices to show that they are independent. Set $x := \sum_{i=1}^n a_i u_i$, $y := \sum_{i=1}^k b_i v_i$ and $z := \sum_{i=1}^l c_i w_i$ and suppose that $x + y + z = 0$. Notice that $x + y \in M$ while $z \in N$. Since $z = -(x + y)$, it follows that $z \in M \cap N$. Thus, there are scalars $d_1, \ldots, d_n$ such that $\sum_{i=1}^l c_i w_i = z = \sum_{i=1}^n d_i u_i$. Since $u_1, \ldots, u_n, w_1, \ldots, w_l$ form a basis for N, this implies that $c_1 = \cdots = c_l = 0 = d_1 = \cdots = d_n$. In particular, this implies that $z = 0$. Thus, $\sum_{i=1}^n a_i u_i + \sum_{i=1}^k b_i v_i = x + y = 0$. The independence of $u_1, \ldots, u_n, v_1, \ldots, v_k$ now implies that $a_1 = \cdots = a_n = 0 = b_1 = \cdots = b_k$. This proves the claim.

□

[3] A minimal spanning set is a spanning set that does not properly contain any other spanning set.