3.3.7 Example If V, W are f.d w/ ordered base $B = (b_1, \dots, b_n)$ and $C = (c_1, \dots, c_m)$, then one can get an ordered basis for $V \otimes W$

$$B \otimes C = (b_i \otimes c_j)_{\substack{1 \le i \le n \\ 1 \le j \le m}}$$

ordered lexicographically. E.g. $n = 3$ $m = 2$

$$B \otimes C = (b_1 \otimes c_1, b_1 \otimes c_2, b_2 \otimes c_1, b_2 \otimes c_2, b_3 \otimes c_1, b_3 \otimes c_2). \square$$

3.3.9 Corollary If V and W are finite-dimensional vector spaces, then

$$\dim_F (V \otimes W) = \dim_F (V) \dim_F (W).$$

$\square$

3.3.9 Proposition If $L_1 : V_1 \longrightarrow W_1$ and $L_2 : V_2 \longrightarrow W_2$ are linear maps, then there is a unique linear map

$$L_1 \otimes L_2 : V_1 \otimes V_2 \longrightarrow W_1 \otimes W_2$$

such that $(L_1 \otimes L_2)(v_1 \otimes v_2) = L_1(v_1) \otimes L_2(v_2)$.

Proof. Homework. $\square$

3.3.10 Proposition Assume the notation of 3.3.9. If B_1, B_2, C_1, C_2 are bases for V_1, V_2, W_1, W_2, respectively, then

$$[L_1 \otimes L_2]_{B_1 \otimes B_2}^{C_1 \otimes C_2} = [L_1]_{B_1}^{C_1} \otimes [L_2]_{B_2}^{C_2}$$

where $\otimes$ on the right is the "Kronecker Product" of Matrices. If

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \in F^{n \times m} \qquad B = \begin{pmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & & \\ b_{p1} & \cdots & b_{pq} \end{pmatrix} \in F^{p \times q},$$

then

$$A \otimes B = \begin{pmatrix} a_{11} B & \cdots & a_{1m} B \\ \vdots & & \\ a_{n1} B & \cdots & a_{nm} B \end{pmatrix} \in F^{np \times mq}.$$

Proof. Exercise. □

3.4 Permutations

3.4.1 Definition The symmetric group of order n is the set S_n of all bijective functions $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$. Elements of S_n are also called permutations. It is a group under function composition.

□

3.4.2 Example (two line notation) One way to write down a permutation σ w/ "two-line notation".

For example,

$$S_4 \ni \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \begin{matrix} \leftarrow \text{domain} \\ \leftarrow \text{range.} \end{matrix}$$

$$\sigma(1)=1, \quad \sigma(2)=3, \quad \sigma(3)=2, \quad \sigma(4)=4$$

□

3.4.3 Definition Let $\sigma \in S_n$. We call σ a **k-cycle** if there exist pairwise distinct elements $a_1, \dots, a_k \in \{1, \dots, n\}$ such that

$$\sigma(a_1)=a_2, \quad \sigma(a_2)=a_3, \quad \dots, \quad \sigma(a_{n-1})=a_n, \quad \sigma(a_n)=a_1$$

In that case, we use **cycle notation**:

$$\sigma = (a_1, \dots, a_n).$$

Two cycles $\sigma = (a_1, \dots, a_n)$ and $\tau = (b_1, \dots, b_m)$ are **disjoint** if the sets $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_m\}$ are disjoint.

□

Remark

(a) Composition is done from right to left, so $\sigma\tau$ means "apply τ first" and then "apply σ".

(b) $(a_1, \dots, a_n) = (a_2, \dots, a_n, a_1) = \dots$

(c) $(a_1, \dots, a_n)^{-1} = (a_n, a_{n-1}, \dots, a_2, a_1)$

(d) Disjoint cycles commute: $\sigma\tau = \tau\sigma$.

(e) A 2-cycle is called a transposition.

3.4.4 Proposition

Every permutation $\sigma \in S_n$ can be written as a product of disjoint cycles.

Proof. Exercise. $\square$

3.4.5 Example

two-line

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 & 6 & 4 \end{pmatrix}$$

cycle notation

$$\pi = (1\,3\,2)(4\,5\,6)$$

↑ ↑

disjoint!

$\square$

3.4.6 Definition

Let $\sigma \in S_n$. A pair $\{i, j\} \subseteq \{1, \dots, n\}$ is call an **inversion** of σ if

$$\frac{j-i}{\sigma(j)-\sigma(i)} < 0.$$

$\square$

3.4.7 Example

(a) $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$

$= (2\,3)$

$\{2, 3\}$ is an inversion

Since $\dfrac{3-2}{\pi(3)-\pi(2)} = \dfrac{3-2}{2-3} < 0$.

(b) $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 3 & 5 & 6 \end{pmatrix}$ $\{2,3\}$ and $\{2,4\}$ are

$= (2\,4\,3)$ inversions.

$\square$

3.4.8 Definition Let $\sigma \in S_n$. Then σ is even if σ has an even number of inversions and odd otherwise. The parity $\operatorname{pty}(\sigma)$ is 0 if σ is even and 1 if σ is odd. The sign $\operatorname{sgn}(\sigma)$ of σ is defined to be

$$\operatorname{sgn}(\sigma) = (-1)^{\operatorname{pty}(\sigma)} = \begin{cases} 1, & \text{if } \sigma \text{ is even} \\ -1, & \text{if } \sigma \text{ is odd} \end{cases}$$

This defines a function $\operatorname{sgn} : S_n \longrightarrow \underbrace{\{1,-1\}}_{\text{a group under mult.}}$. $\square$

3.4.9 Example A transposition is a permutation that swaps two elements, i.e., a 2-cycle (i,j) for some $i,j \in \{1,\dots,n\}$ w/ $i<j$. One can show that (i,j) has $2(j-i)-1$ inversions. Since this is an odd number, 2-cycles are odd and $\operatorname{sgn}(i,j) = -1$.

$\square$

3.4.10 Lemma Let $\sigma, \tau \in S_n$. Then

$$\operatorname{sgn}(\sigma) = \prod_{i<j} \frac{\tau(j)-\tau(i)}{\sigma(\tau(j))-\sigma(\tau(i))}.$$

Proof. Since τ is a bijection the following function

set of all 2 elmt subsets of $\{1,\dots,n\}$

$$\Phi_\tau : \binom{\{1,\dots,n\}}{2} \longrightarrow \binom{\{1,\dots,n\}}{2}$$
$$\{i,j\} \longmapsto \{\tau(i),\tau(j)\}$$

is surjective and hence bijective. So

$$\prod_{i<j} \frac{\tau(j)-\tau(i)}{\sigma(\tau(j))-\sigma(\tau(i))} = \prod_{i<j} \frac{j-i}{\sigma(j)-\sigma(i)}$$

Since Φ_τ is bijective. Since Φ_σ is bijective the numerator and denominator are the same, except possible the signes. So the product is 1 or -1. But $\frac{j-i}{\sigma(j)-\sigma(i)}$ is negative iff $\{i,j\}$ is is an inversion. Hence, the product is 1 iff the number of inversions is even. $\square$

3.4.11 Example

Consider $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. Then

$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$.

$$\prod_{i<j} \frac{j-i}{\sigma(j)-\sigma(i)} = \frac{(2-1)(3-1)(3-2)}{(1-3)(2-3)(2-1)} = 1 = \operatorname{sgn}(\sigma)$$

$$\prod_{i<j} \frac{\tau(j)-\tau(i)}{\sigma\tau(j)-\sigma\tau(i)} = \frac{(1-2)(3-2)(3-1)}{(3-1)(2-1)(2-3)} = 1 .$$

□

3.4.12 Proposition If $\sigma, \tau \in S_n$, then

$$\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau).$$

That is, $\operatorname{sgn} : S_n \longrightarrow \{1,-1\}$ is a group homomorphism.

Proof.

$$\operatorname{sgn}(\sigma\tau) = \prod_{i<j} \frac{j-i}{\sigma\tau(j)-\sigma\tau(i)}$$

$$= \prod_{i<j} \frac{\tau(j)-\tau(i)}{\sigma\tau(j)-\sigma\tau(i)} \cdot \prod_{i<j} \frac{j-i}{\tau(j)-\tau(i)}$$

$$= \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau).$$

□

3.4.13 Proposition Every permutation $\sigma \in S_n$ can be written as a product of transpositions. Even a product of adjacent transpositions $(1,2), (2,3), \dots, (n-1,n)$.

Proof. By Prop 3.4.4, it suffices to prove the claim for a cycle $\sigma = (a_1, \dots, a_n)$. We have

$$\sigma = (a_1, a_n) \cdots \cdots (a_1, a_3)(a_1, a_2).$$

Adjacent transpositions: exercise.

□

3.4.14 Corollary If σ is a product of k-transpositions, then $\operatorname{pty}(\sigma) = k \pmod 2$. □

3.5 Alternating Multilinear Forms

3.5.1 Definition An n-form $\omega : V \times \cdots \times V \to F$ is called alternating if

$$\omega(v_1, \dots, v_n) = 0 \quad \text{whenever } v_i = v_j \text{ for some } i \neq j$$

The set of all alternating n-forms is a subspace $\Lambda^n(V) \subseteq T^n(V)$. $\square$

3.5.2 Proposition Let $\omega \in \Lambda^n(V)$. Then

(a) $\omega(v_1, \dots, v_{i+1}, v_i, \dots, v_n) = -\omega(v_1, \dots, v_n)$

(b) For any $\sigma \in S_n$

$$\omega(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = \operatorname{sgn}(\sigma)\, \omega(v_1, \dots, v_n)$$

(c) $\omega(v_1, \dots, v_i + \alpha v_j, \dots, v_n) = \omega(v_1, \dots, v_i, \dots, v_n)$ for $i \neq j$.

Proof.

(a) $\omega(v_1, \dots, v_{i+1}, v_i, \dots, v_n) + \omega(v_1, \dots, v_n)$

$= \omega(v_1, \dots, v_{i+1} + v_i, v_i + v_{i+1}, \dots, v_n)$

$= 0$

since w is alternating.

(b) Induction on the number of transpositions in a decomp of σ into a product of adjacent transpositions $\sigma = \tau_1 \cdots \tau_k$. Then (a) proves the case $k=1$. Assume the statement holds for $k-1$. Then

$$\begin{aligned} \operatorname{sgn}(\sigma)\, w(v_1, \dots, v_n) &= \operatorname{sgn}(\tau_1 \cdots \tau_k)\, w(v_1, \dots, v_n) \\ &= \operatorname{sgn}(\tau_1 \cdots \tau_{k-1}) \operatorname{sgn}(\tau_k)\, w(v_1, \dots, v_n) \\ &= \operatorname{sgn}(\tau_k) \operatorname{sgn}(\tau_1 \cdots \tau_{k-1})\, w(v_1, \dots, v_n) \\ &= \operatorname{sgn}(\tau_k)\, w(v_{\tau_1 \cdots \tau_{k-1}(1)}, \dots, v_{\tau_1 \cdots \tau_{k-1}(1)}) \\ &= W(\tau_{\sigma(1)}, \dots, \tau_{\sigma(n)}). \end{aligned}$$

(c)
$$\begin{aligned} & w(v_1, \dots, v_i + \alpha v_j, \dots, v_n) \\ &= w(v_1, \dots, v_i, \dots, v_n) + \alpha w(v_1, \dots, \overset{i\text{th entry}}{v_j}, \dots, v_n) \\ &= w(v_1, \dots, v_n) + 0 \end{aligned}$$

Since $i \neq j$ implies $w(v_1, \dots, v_j, \dots, v_n) = 0$.

$\square$

3.5.3 Proposition Suppose $v_1, \dots, v_n \in V$ are dependent. Then

$$w(v_1, \dots, v_n) = 0$$

for any $w \in \Lambda^n(V)$.

Proof. There is a nontrivial linear combination

$$v_j = \sum_{i \neq j} \alpha_i v_i .$$

Then for any $\omega \in \Lambda^n(V)$, we have

$$\begin{aligned} \omega(v_1, \ldots, v_j, \ldots, v_n) &= \omega\Big(v_1, \ldots, \sum_{i \neq j} \alpha_i v_i, \ldots, v_n\Big) \\ &= \sum_{i \neq j} \alpha_i \, \omega(v_1, \ldots, v_i, \ldots, v_n) \\ &= 0 . \end{aligned}$$

□

3.5.4 Corollary If V is n-dimensional and $k > n$, then the only alternating k-form is the zero form.

Proof. Any collection of k vectors in V will be dependent since $k > \dim V$. Apply 3.5.3. □

3.5.5 Proposition If V is n-dimensional, then

$$\dim \Lambda^k(V) \leq \binom{n}{k} .$$

Proof. Let $\omega \in \Lambda^k(V)$. Then there is a unique linear map $\overline{\omega} : V \otimes \cdots \otimes V \longrightarrow F$ corresponding to ω. Let $(b_1, \ldots, b_n)$ be a basis for V. Then

$$\{ b_{i_1} \otimes \cdots \otimes b_{i_k} \mid 1 \leq i_1 \leq \cdots \leq i_k \leq n \}$$

is a basis for $\underbrace{V \otimes \cdots \otimes V}$. Then $\overline{\omega}$ is

k - times.

uniquely determined by its values on the basis. So ω is determined by the values

$$\omega(b_{i_1}, \dots, b_{i_k}) \qquad 1 \leq i_1, \dots, i_k \leq n.$$

Since this is zero whenever two indices are the same, we need only consider indices $i_1, \dots, i_k$ that are pairwise distinct. By 3.5.2(b), we need only consider the values

$$\omega(b_{i_1}, \dots, b_{i_k})$$

where $1 \leq i_1 < \cdots < i_k \leq n$. The number of possibilities for the indices is $\binom{n}{k}$.

□

3.5.6 Corollary If $\dim V = n$, then $\dim \Lambda^n(V) \leq 1$.

□

3.5.7 Theorem If $\dim V = n$, then $\dim \Lambda^k(V) = \binom{n}{k}$.

Proof. Let $B = (b_1, \dots, b_n)$ be a basis for V and let $B^* = (\varphi_1, \dots, \varphi_n)$ be the dual basis. Let $\mathcal{I}$ denote the set of increasing functions from $[k] = \{1, \dots, k\}$ and $[n] = \{1, \dots, n\}$. Note: $|\mathcal{I}| = \binom{n}{k}$. For each $g \in \mathcal{I}$, define $\omega_g : V \times \cdots \times V \to F$ via

$$\omega_g(v_1, \dots, v_k) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} \varphi_{g(\sigma(i))}(v_i).$$

Exercise: show that W_g is an alternating k-form.

I will prove that $\{W_g \mid g \in \tilde{I}\}$ spans $\Lambda^k(V)$.

Let $f \in \Lambda^k(V)$. Then (writing $v_j = \sum_{l_j=1}^{n} \varphi_{l_j}(v_j)\, b_{l_j}$)

$$f(v_1, \dots, v_k) = \sum_{l_1=1}^{n} \sum_{l_2=1}^{n} \cdots \sum_{l_k=1}^{n} \left(\prod_{i=1}^{k} \varphi_{l_i}(v_i) \right) f(b_{l_1}, \dots, b_{l_k})$$

$$= \sum_{\substack{(l_1, \dots, l_k) \\ \in [n] \times \cdots [n]}} \prod_{i=1}^{k} \varphi_{l_i}(v_i) \; f(b_{l_1}, \dots, b_{l_k})$$

There is a bijection

$$[n] \times \cdots \times [n] \longrightarrow \mathrm{Fun}([k],[n])$$
$$(l_1, \dots, l_k) \longmapsto l(i) := l_i$$

Given $l \in \mathrm{Fun}([k],[n])$, write $\alpha_l := \prod_{i=1}^{k} \varphi_{l(i)}(v_i)$

and $b_l := (b_{l(1)}, \dots, b_{l(k)})$. Then

$$= \sum_{l \in \mathrm{Fun}([k],[n])} \alpha_l \, f(b_l)$$

$$= \sum_{l \in \mathrm{Inj}(\mathrm{Fun}([k],[n]))} \alpha_l \, f(b_l) \qquad \left(\begin{array}{l} f \text{ is alternating} \\ \text{so } f(b_l) = 0 \\ \text{if } l \text{ is not} \\ \quad \text{injective} \end{array} \right)$$

$$= \sum_{g \in \tilde{I}} \sum_{\sigma \in S_k} \alpha_{g \circ \sigma} \, f(b_{g \circ \sigma}) \qquad \left(\begin{array}{l} \mathrm{Inj}(\mathrm{Fun}([k],[n])) \\ = \{ g \circ \sigma \mid g \in \tilde{I}, \sigma \in S_k \} \end{array} \right)$$

$$= \sum_{g \in \tilde{I}} f(b_g) \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \, \alpha_{g \circ \sigma}$$

$$= \sum_{g \in \tilde{I}} f(b_g) \, W_g .$$

This proves that $\{\omega_g \mid g \in \mathcal{I}\}$ spans. But it has $\binom{n}{k}$ elements and $\dim \Lambda^k(V) \leq \binom{n}{k}$. Thus, it forms a basis and $\dim \Lambda^k(V) = \binom{n}{k}$.

$\square$

3.5.8 Corollary There is a unique alternating n-form

$$\det : F^{n\times n} = F^n \times \cdots \times F^n \longrightarrow F$$

which satisfies $\det(I_n) = 1$. We call it the determinant on $F^{n\times n}$.

Proof. By 3.5.7, $\dim \Lambda^n(F^n) = \binom{n}{n} = 1$. Note that $\mathcal{I}$ contains only the identity function $\mathrm{id} : [n] \to [n]$. So a basis for $\Lambda^n(F^n)$ is $\det := \omega_{\mathrm{id}}$. We have

$$\det(v_1, \ldots, v_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \varphi_{\sigma(i)}(v_i)$$

where $\varphi_1, \ldots, \varphi_n$ is dual to the standard basis $e_1, \ldots, e_n$ for F^n. Since $I_n = (e_1 | \cdots | e_n)$, we have

$$\begin{aligned}\det(I_n) &= \det(e_1, \ldots, e_n) \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \varphi_{\sigma(i)}(e_i). \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \delta_{\sigma(1),1}\, \delta_{\sigma(2),1} \cdots \delta_{\sigma(n)n} \quad \left(\begin{array}{c} = 0 \text{ unless} \\ \sigma(i) = i \\ \forall i \end{array}\right)\end{aligned}$$

$$= \operatorname{sgn}(\mathrm{id})\, \delta_{1,1}\, \delta_{22} \cdots \delta_{nn}$$
$$= 1.$$

Existence is proved. If $\omega \in \Lambda^n(F^n)$ s.t. $\omega(I_n) = 1$, then $\omega = \lambda \det$ for some $\lambda \in F$. Then $\lambda = \lambda \det(I_n) = \omega(I_n) = 1$. Uniqueness proved. □

3.5.9 Corollary ("Leibniz Formula" for determinant)

Let $A = (a_{ij}) \in F^{n \times n}$. Then

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i}$$

Proof. Suppose

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

Then

$$f_{\sigma(i)} \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix} = f_{\sigma(i)} \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{bmatrix}_{\mathcal{E}}$$
$$= a_{\sigma(i)i}.$$

□

3.5.10 Example

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \sum_{\sigma \in S_2} \operatorname{sgn}(\sigma) \prod_{i=1}^{2} a_{\sigma(i)i}$$
$$= \operatorname{sgn}(\mathrm{id})\, a_{11} a_{22} + \operatorname{sgn}(12)\, a_{21} a_{12}$$
$$= a_{11} a_{22} - a_{21} a_{22}.$$

□

3.5.11 Proposition Let $A \in F^{n\times n}$. Then $\det(A) = \det(A^T)$ and $\det : F^{n\times n} \longrightarrow F$ is an alternating multilinear map on the rows of $F^{n\times n}$.

Proof. Viewing det as a function on the the rows of $F^{n\times n}$ amounts to precomposing det w/ the map $(-)^T : F^{n\times n} \longrightarrow F^{n\times n}$. Write $A = (a_{ij})$. Then $A^T = (a_{ji})$. Hence,

$$\begin{aligned}
\det(A^T) &= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \\
&= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma^{-1}(i),\sigma(\sigma^{-1}(i))} \quad \left(\sigma \text{ is bijective}\right) \\
&= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma^{-1}(i),i} \\
&= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i)i} \quad \left(\begin{array}{l} S_n \longrightarrow S_n \\ \sigma \longmapsto \sigma^{-1} \\ \text{is a bijection} \end{array}\right) \\
&= \det(A).
\end{aligned}$$

□

3.5.12 Proposition Let $A \in F^{n\times n}$.

(a) Suppose B is obtained from A via $C_i \longleftarrow \alpha C_i$ or $R_i \longleftarrow \alpha R_i$, $\alpha \neq 0$. Then $\det(B) = \alpha \det(A)$.

(b) Suppose B is obtained from A via $C_i \longleftrightarrow C_j$ or $R_i \longleftrightarrow R_j$, $i \neq j$. Then $\det(B) = -\det(A)$.

(c) Suppose B is obtained from A via $C_i \leftarrow C_i + \alpha C_j$ or $R_i \leftarrow R_i + \alpha R_j$, $i \neq j$. Then $\det(B) = \det(A)$.

Proof. Statements about column operations follow from Def 3.3.1, Prop 3.5.2(a), and Prop 3.5.2(c). Statements about row operations follow from Prop 3.5.11. $\square$

3.5.13 Proposition Let $A \in F^{n \times n}$. The following are equivalent:

(a) $\det(A) \neq 0$

(b) the columns of A are independent in F^n

(c) the rows of A are independent in F^n

(d) A is invertible

(e) $\mathrm{Rref}(A) = I_n$

(f) A is a product of elementary matrices

Proof. Equivalence of (b) - (f) can be deduced from section 2.9. We prove (a) $\Leftrightarrow$ (b).

(a) $\Rightarrow$ (b) Follows from contrapositive of Prop 3.5.3.

(b) $\Rightarrow$ (a) The columns $c_1, \dots, c_n$ of A form a basis for F^n. Write $[e_i]_{(c_1,\dots,c_n)} = (\beta_{1j}, \dots, \beta_{nj})$.

Then

$$1 = \det(e_1, \dots, e_n) = \det\left(\sum_{i_1=1}^{n} \beta_{i_1 1} c_{i_1}, \sum_{i_2=1}^{n} \beta_{i_2 2} c_{i_2}, \dots\right)$$

$$= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} \left(\prod_{j=1}^{n} \beta_{ij}\right) \det(c_{i_1}, \dots, c_{i_n})$$

This proves $\det(c_{i_1}, \dots, c_{i_n}) \neq 0$. Thus, $i_1, \dots, i_n$ are all distinct. There exists $\sigma \in S_n$ satisfying $\sigma(e_j) = j$. Thus,

$$\operatorname{sgn}(\sigma) \det(A) = \det(c_{i_1}, \dots, c_{i_n}) \neq 0.$$

☐

3.5.14 Proposition Let $A, B \in F^{n \times n}$. Then $\det(AB) = \det(A)\det(B)$. In particular, if A is invertible, then $\det(A^{-1}) = \det(A)^{-1}$.

Proof. If A is invertible, then

$$1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1}).$$

Thus, $\det(A^{-1}) = 1/\det(A)$. So it suffices to prove that det multiplicative. Exercise using elementary.

☐

Remark: By 3.5.14, similar matrices have the same determinant:

$$\det(B) = \det(P^{-1}AP)$$
$$= \det(P)^{-1}\det(A)\det(P)$$

$$= \det(A).$$

3.5.15 Definition The determinant of a linear operator $L : V \longrightarrow V$ on a finite-dimensional vector space V is defined to be

$$\det(L) := \det([L]_B)$$

where B is any basis for V. This is well-defined by 3.5.14. □

3.5.16 Proposition If $A \in F^{k\times k}$ and $B^{(n-k)\times(n-k)}$, $n>k$, then

$$\det(A\oplus B) = \det(A)\det(B).$$

Here $A\oplus B$ is the $n\times n$ "block diagonal" matrix

$$A\oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Proof. Write $A=(a_{ij})$, $B=(b_{ij})$, and $C=(c_{ij})$. Define

$$T_n := \left\{ \sigma\in S_n \mid \sigma(\{1,\dots,k\}) \subseteq \{1,\dots,k\}, \sigma(\{k+1,\dots,n\} \subseteq \{k+1,\dots,n\} \right\}$$

Then $c_{i,\sigma(i)} = 0$ whenever $\sigma \notin T_n$. thus,

$$\det(A\oplus B) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} c_{i,\sigma(i)}$$

$$= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} c_{i\sigma(i)} \prod_{i=k+1}^{n} c_{i\sigma(i)}$$

$$= \sum_{p \in S_k} \sum_{\tau \in S_{n-k}} \operatorname{sgn}(p\tau) \prod_{i=1}^{k} c_{i,p\tau(i)} \prod_{i=k+1}^{n} c_{i,p\tau(i)}$$

$$= \sum_{p \in S_k} \operatorname{sgn}(p) \prod_{i=1}^{k} a_{i,p(i)} \sum_{\tau \in S_{n-k}} \operatorname{sgn}(\tau) \prod_{i=1}^{n-k} b_{i,\tau(i)}$$

$$= \det(A)\det(B). \qquad \square$$

More generally, $\det\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \det(A)\det(B).$

$\det\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} = \det(A)\det(B)$

However, in general, $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} \neq \det(A)\det(D).$

Chapter IV: Canonical Forms

4.1 Introduction

Goal: Given a linear operator $L: V \to V$ on a f.d. vector space, we want to find an ordered basis $\mathcal{B}$ for V such that $[L]_{\mathcal{B}}$ is as simple as possible.

Motivating Example

What are the simplest matrices one can think of?

- Identity matrix $I_n \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$
- Scalar multiple of identity $\lambda I_n = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$

- diagonal matrices $\operatorname{diag}(\lambda_1, \ldots, \lambda_n) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

Suppose $\mathcal{B}$ is a basis for V such

$$[L]_{\mathcal{B}} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

Then we already know a lot about L:

(a) Geometrically, L is expanding or contracting in the direction of each basis vector

$$L(b_i) = \lambda_i b_i \quad , \qquad i = 1, \ldots, n$$

(b) Certain invariants of L can be read immediately from the matrix

- $\det(L) = \prod_{i=1}^{n} \lambda_i$.

- $\operatorname{tr}(L) = \sum_{i=1}^{n} \lambda_i$

Questions:

- Can every linear operator be represented by a diagonal matrix?
- If not, for operators does such a basis exist?
- How can we compute a basis, if it is exists?
- If no such basis exists, can we find a different kind of "simple" matrix to represent L?

To answer these questions, we will the algebra of linear operators.