3.3.1 Definition Let $V_1, \dots, V_n$ and W be vector spaces over F. A **multilinear map** is a function

$$L: V_1 \times \cdots \times V_n \longrightarrow W$$

that is linear in each "coordinate":

$$L(v_1, \dots, \alpha v_i + \beta v_i', \dots, v_n) = \alpha L(v_1, \dots, v_i, \dots, v_n) + \beta L(v_1, \dots, v_i', \dots, v_n)$$

for all $i = 1, \dots, n$.

Equivalently, for each $i = 1, \dots, n$, the composition

$$V_i \xrightarrow{\subseteq} V_1 \times \cdots \times V_n \xrightarrow{L} W$$

is a linear map. The set of all multilinear maps from $V_1 \times \cdots \times V_n$ to W is denoted by

$$\mathrm{Mult}(V_1 \times \cdots \times V_n, W).$$

□

Remark: In the definition above, we are thinking of $V_1 \times \cdots \times V_n$ as a set and not a vector space.

Remark: $\mathrm{Mult}(V_1 \times \cdots \times V_n, W)$ is a subspace of $\mathrm{Maps}(V_1 \times \cdots \times V_n, W)$.

Remark: Special cases:

- $(n=1)$ just linear maps $V \longrightarrow W$ $\quad \mathrm{Mult}(V, W) = \mathrm{Hom}(V, W)$
- $(n=2)$ called bilinear maps $\quad \mathrm{Bil}(V_1 \times V_2, W) = \mathrm{Mult}(V_1 \times V_2, W)$

$$V_1 \times V_2 \longrightarrow W$$

- For a vector space V, a multilinear map

$$\underbrace{V \times \cdots \times V}_{\substack{n\text{-copies} \\ \text{of } V}} \longrightarrow F$$

is called a multilinear n-form (or just n-form). The vector space of n-forms on V is denoted by

$$\mathcal{T}^n(V) := \mathrm{Mult}(V \times \cdots \times V, F).$$

1-forms are just linear functionals $\mathcal{T}^1(V) = V^*$

2-forms are called bilinear forms

$$\square$$

Remark: The idea of a multilinear is a natural generalization of a linear map. But it would be kind of a bummer to build a theory of multilinear algebra from first principles. Fortunately, there is a "universal" method to turn multilinear maps into linear maps on a special vector space called the "tensor product". The process will be invertible, thus multilinear algebra will be subsumed by linear algebra.

Idea: (for bilinear maps) Given two vector spaces V and W, the tensor product $V \otimes W$ will be a new vector space equipped with a special bilinear map

$$\begin{tikzcd} V \times W \arrow[r, "-\otimes-"] & V \otimes W \\ (v,w) \arrow[r, mapsto] & v \otimes w \end{tikzcd}$$

The bilinear map $-\otimes-$ will be special in the sense that it is the "universal" bilinear map defined on $V \times W$. Precisely, this means that given any vector Z and and bilinear map $B: V \times W \longrightarrow Z$, there exists a unique linear map $L: V \otimes W \longrightarrow Z$ such that $L(-\otimes-) = B$. In other words, there exists a unique linear map such that the diagram

$$\begin{tikzcd} V \times W \arrow[r, "B"] \arrow[d, "-\otimes-"'] & Z \\ V \otimes W \arrow[ur, dashed, "\exists! L"'] & \end{tikzcd}$$

commutes. This is the "Universal Property of the Tensor Product". By defining $V \otimes W$ so that it has this property, we will obtain a bijection

$$\begin{tikzcd} \mathrm{Bil}(V \times W, Z) \arrow[r, "\sim"] & \mathrm{Hom}(V \otimes W, Z) \\ B \arrow[r, mapsto, "\exists!"] & L \\ L(-\otimes-) & L \arrow[l, mapsto] \end{tikzcd}$$

Recall: We have seen several other universal properties.

- U.P. of the Free Vector Space $F^{(X)}$ w/ basis X

$$\mathrm{Hom}(F^{(X)}, Z) \xrightarrow{\sim} \mathrm{Maps}(X, Z)$$

- U.P. of the Quotient space

$$\mathrm{Hom}(V/W, Z) \xrightarrow{\sim} \{ L \in \mathrm{Hom}(V,Z) \mid W \subseteq \ker(L) \}$$

- U.P. of the Direct Sum

$$\mathrm{Hom}(V \oplus W, Z) \xrightarrow{\sim} \mathrm{Hom}(V,Z) \oplus \mathrm{Hom}(W,Z)$$

3.3.2 Definition Let V and W be vector spaces over F. The tensor product $V \otimes W$ of V and W is the F-vectorspace defined as follows:

$$V \otimes W := F^{(V \times W)} \big/ N$$

where

- $F^{(V \times W)}$ is the free vector space w/ basis $V \times W$

$$F^{(V \times W)} = \left\{ f : V \times W \to F \;\middle|\; \begin{array}{l} f(v,w) = 0 \ \forall \text{ but finitely} \\ \text{many } (v,w) \in V \times W \end{array} \right\}$$

$$= \bigoplus_{(v,w) \in V \times W} F$$

An element $x \in F^{(V \times W)}$ is usually written as a formal linear combination

$$x = \sum_{(v,w) \in V \times W}' \alpha_{(v,w)} (v,w)$$

where $\alpha_{(v,w)} = 0$ for all but finitely many $(v,w) \in V \times W$. Addition and scalar mult of formal sums is done in the obvious way.

- $$N := \operatorname{span}\Big(\{ (v_1+v_2, w) - (v_1,w) - (v_2,w) \mid v_1, v_2 \in V, w \in W \}$$
$$\cup \{ (v, w_1 + w_2) - (v, w_1) - (v, w_2) \mid v \in V, w_1, w_2 \in W \}$$
$$\cup \{ (\alpha v, w) - \alpha(v,w) \mid \alpha \in F, v \in V, w \in W \}$$
$$\cup \{ (v, \alpha w) - \alpha(v,w) \mid \alpha \in F, v \in V, w \in W \} \Big)$$

Automatically, $V \otimes W$ is a vector space. It comes w/ a map

$$\begin{tikzcd}
\underline{V \times W} \arrow[r, hook, "\subseteq"] & F^{(V\times W)} \arrow[r, two heads, "\pi"] & F^{(V\times W)}/N =: V \otimes W \\
(v,w) \arrow[r, mapsto] & (v,w) \arrow[r, mapsto] & (v,w) + N =: v \otimes w
\end{tikzcd}$$

which we denote by $- \otimes - : V \times W \longrightarrow V \otimes W$
$$(v,w) \longmapsto v \otimes w$$

In other words, $v \otimes w := (v,w) + N$ is just the coset w/ representative (v,w). The coset $v \otimes w$ is called a simple tensor.

□

3.3.3 Proposition Let V and W be vector spaces over F. For all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, $\alpha \in F$, we have

(a) $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$

(b) $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$

(c) $(\alpha v) \otimes w = \alpha (v \otimes w) = v \otimes (\alpha w)$.

In other words, $- \otimes - : V \times W \to V \otimes W$ is bilinear.

Proof. (a) $(v_1 + v_2) \otimes w = (v_1 + v_2, w) + N$

$$= (v_1+v_2, w) + \underbrace{(v_1, w) + (v_2, w) - (v_1 + v_2, w)}_{\in N} + N$$

$$= (v_1, w) + (v_2, w) + N$$

$$= (v_1, w) + N + (v_2, w) + N$$

$$= v_1 \otimes w + v_2 \otimes w.$$

Others are similar.

□

Remark: $F^{(V \times W)}$ is the vector space w/ basis $V \times W$. An arbitrary element of $F^{(V \times W)}$ is a formal linear combination $f \in F^{(V \times W)}$

$$x = \sum_{(v,w) \in V \times W} \alpha_{(v,w)} (v,w)$$

of elements of $V \times W$ w/ finitely many nonzero coefficients. Passing the quotient yields an arbitrary element of $V \otimes W$

$$x + N = \left(\sum_{(v,w) \in V \times W} \alpha_{(v,w)} (v,w) \right) + N$$

$$= \sum_{(v,w) \in V \times W} \left(\alpha_{(v,w)} (v,w) + N \right)$$

$v' := \alpha_{(v,w)} v$ $$= \sum_{(v,w)\in V\times W}' \left((\alpha_{(v,w)} v, w) + N \right)$$

$$= \sum_{(v',w)\in V\times W}' \left((v',w) + N \right) = \sum' v' \otimes w.$$

Thus, an arbitrary element of $V \otimes W$ is just a finite sum of simple tensors

$$\sum_{i=1}^{n} v_i \otimes w_i \,, \qquad v_i \in V \,,\ w_i \in W.$$

In particular, the set $\{ v\otimes w \mid v \in V, w\in W\}$ is a spanning set for $V \otimes W$.

3.3.4 Theorem (Universal Property of $\otimes$)

Let V, W be vector spaces over F. For any vector space Z and any bilinear map $B: V\times W \to Z$, there is a unique linear map $L: V\otimes W \to Z$ with the property that $L(v\otimes w) = B(v,w)$ for all $v \in V$ and $w \in W$. In other words, there is a unique linear map $L: V\otimes W \to Z$ making the diagram

$$\begin{tikzcd}
V \times W \arrow[r, "B"] \arrow[d, "-\otimes-"'] \arrow[dr, phantom, "\circlearrowleft"] & Z \\
V \otimes W \arrow[ur, dashed, "\exists ! L"'] &
\end{tikzcd}$$

commute.

Proof. (existence) We define $L: V\otimes W \longrightarrow Z$. First, use the Universal Property of $F^{(V\times W)}$. We have a function $V\times W \xrightarrow{B} Z$. There exists a unique linear map $K: F^{(V\times W)} \longrightarrow Z$ such that the diagram

$$\begin{tikzcd}
V\times W \arrow[r, "B"] \arrow[d, "i"'] \arrow[dr, phantom, "\circlearrowleft"] & Z \\
F^{(V\times W)} \arrow[ur, dashed, "\exists! K"'] &
\end{tikzcd}$$

Now invoke the Universal Property of the Quotient $F^{(V\times W)}/N$. Since B is bilinear, $N \subseteq \ker(K)$. There exists a unique linear map making the diagram

$$\begin{tikzcd}
F^{(V\times W)} \arrow[r, "K"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & Z \\
V\otimes W \arrow[ur, dashed, "\exists! L"'] &
\end{tikzcd}$$

commute. Combine both diagrams

$$\begin{tikzcd}
V\times W \arrow[r, "B"] \arrow[d, "i"'] \arrow[dr, phantom, "\circlearrowleft"] & Z \\
F^{(V\times W)} \arrow[ur, dashed, "K"'] \arrow[d, "\pi"'] \arrow[r, phantom, "\circlearrowleft"] & \\
V\otimes W \arrow[uur, dashed, "L"'] &
\end{tikzcd}$$

The interior triangles commute. Therefore, the outer triangle

$$V\times W \xrightarrow{B} Z$$

$$\begin{tikzcd} {} \arrow[d, "-\otimes-"'] & G \\ V\otimes W \arrow[ur, "L"'] & \end{tikzcd}$$

commutes. This proves existence.

(uniqueness) Suppose $L' : V\otimes W \longrightarrow Z$ also satisfies $L'(-\otimes-) = B$. For any simple tensor $v\otimes w \in V\otimes W$, we have

$$L'(v\otimes w) = B(v,w) = L(v\otimes w).$$

Since L, L' are linear and the simple tensors span, this implies $L' = L$.

$\square$

Remark: The Universal Property of the Tensor Product establishes a bijection: for any vector space Z,

$$\begin{array}{ccc} \mathrm{Bil}(V\times W, Z) & \xrightarrow{\sim} & \mathrm{Hom}(V\otimes W, Z) \\ B & \longmapsto & L \\ L(-\otimes-) & \longleftarrow\!\shortmid & L \end{array}$$

In fact, this is an isomorphism of vector spaces.

Remark: For vector spaces $V_1, \dots, V_n$, one can define

$$V_1 \otimes \cdots \otimes V_n$$

either recursively $V_1\otimes\cdots\otimes V_n := ((V_1\otimes V_2)\otimes V_3)\cdots\cdots)\otimes V_n$ or as a suitable quotient of $F^{(V_1\times\cdots\times V_n)}$. One can show that both constructions are isomorphic and that the tensor product is "associative" so that the recursive construction is independent of the bracketing.

It comes w/ a universal multilinear map

$$- \otimes - \cdots - \otimes - : V_1 \times \cdots \times V_n \longrightarrow V_1 \otimes \cdots \otimes V_n$$

and satisfies a universal property which establishes a bijection

$$\mathrm{Mult}(V_1 \times \cdots \times V_n , Z) \xrightarrow{\sim} \mathrm{Hom}(V_1 \otimes \cdots \otimes V_n , Z)$$

for any vector space Z. This is actually an isomorphism of vector spaces. $\square$

3.3.5 Example

(a) $$T^n(V) := \mathrm{Mult}(V \times \cdots \times V, F) \cong \mathrm{Hom}(V \otimes \cdots \otimes V ; F)$$
$$\cong (V \otimes \cdots \otimes V)^*$$

A multilinear n-form corresponds to a linear functional on $V \otimes \cdots \otimes V$.

(b) $$\mathrm{Hom}(V, V^*) \cong (V \otimes V)^*$$

I claim these two spaces are isomorphic. Define a map

$$\mathrm{Hom}(V, V^*) \xrightarrow{\Phi} (V \otimes V)^*$$

as follows. Given $K \in \mathrm{Hom}(V, V^*)$, we have a bilinear map

$$V \times V \longrightarrow F$$
$$(v, w) \longmapsto (K(v))(w)$$

By UP of Tensor Product, there is a unique linear map

$$\Phi(K) : V \otimes V \longrightarrow F$$

satisfying $(\Phi(K))(v \otimes w) = (K(v))(w)$.

Exercise: prove this map is an isomorphism.

$\square$

3.3.6 Theorem Let V and W be vector spaces w/ bases $B = \{b_i\}_{i \in I}$ and $C = \{c_j\}_{j \in J}$ where I and J are any sets. Then

$$B \otimes C := \{ b_i \otimes c_j \mid (i,j) \in I \times J \}$$

is a basis for $V \otimes W$.

Proof. (spans) Suffices to show that every simple tensor $v \otimes w$ is in the span of $B \otimes C$. Let $v \in V$ and $w \in W$. Then $v = \sum_{i \in I'} \alpha_i b_i$ and $w = \sum_{j \in J'} \beta_j c_j$

for some finite subsets $I' \subseteq I$ and $J' \subseteq J$.

$$\begin{aligned} v \otimes w &= \Big(\sum_{i \in I'} \alpha_i b_i \Big) \otimes \Big(\sum_{j \in J'} \beta_j c_j \Big) \\ &= \sum_{i \in I'} \alpha_i \Big(b_i \otimes \sum_{j \in J'} \beta_j c_j \Big) \\ &= \sum_{i \in I'} \alpha_i \sum_{j \in J'} \beta_j (b_i \otimes c_j) \\ &= \sum_{(i,j) \in I' \times J'} \alpha_i \beta_j (b_j \otimes c_j) \in \operatorname{Span}(B \otimes C). \end{aligned}$$

(independence) Let $B^* = \{\delta_i\}_{i \in I}$ be the set of linear functionals "dual" to B. I.e. $\varphi_i : V \longrightarrow F$ is the unique linear map satisfying

$$\varphi_i(b_j) = \delta_{ij}.$$

Similarly, define $C^* = \{\psi_j\}_{j\in J}$. For each (i,j), we have a bilinear map

$$\begin{aligned} V\times W &\longrightarrow F \\ (v,w) &\longmapsto \varphi_i(v)\cdot\psi_j(w) \end{aligned}$$

By UP of Tensor Product, there is a unique linear map $\varphi_i\otimes\psi_j : V\otimes W \longrightarrow F$ satisfying $(\varphi_i\otimes\psi_j)(v\otimes w) = \varphi_i(v)\psi_j(w)$. Suppose

$$\sum_{(i,j)\in I'\times J'} \alpha_{ij}\,(b_i\otimes c_j) = 0.$$

for some finite subsets $I'\subseteq I$ and $J'\subseteq J$. Fix $(i_0,j_0)\in I'$. Then

$$\begin{aligned} 0 &= (\varphi_{i_0}\otimes\psi_{j_0})(0) \\ &= (\varphi_{i_0}\otimes\psi_{j_0})\Big(\sum_{(i,j)\in I'\times J'}\alpha_{ij}\,(b_i\otimes c_j)\Big) \\ &= \sum_{(i,j)\in I'\times J'}\alpha_{ij}(\varphi_{i_0}\otimes\psi_{j_0})(b_i\otimes c_j) \\ &= \sum_{(i,j)\in I'\times J'}\alpha_{ij}\,\varphi_{i_0}(b_i)\,\psi_{j_0}(c_j) \\ &= \sum_{(i,j)\in I'\times J'}\alpha_{ij}\,\delta_{i_0,i}\,\delta_{j_0,j} \\ &= \alpha_{i_0 j_0}. \end{aligned}$$

This completes the proof.

$\square$