Chapter III: Duality and Multilinear Algebra

3.1 Linear Functionals / Dual Space

3.1.1 Definition

A linear functional on a vector space V over F is just a linear map $f: V \longrightarrow F$. $\square$

3.1.2 Example

(a) The trace of a matrix defines a linear function

$$\begin{aligned} tr: F^{n\times n} &\longrightarrow F \\ (a_{ij}) &\longmapsto \sum_{i=1}^{n} a_{ii} \end{aligned}$$

(b) Evaluating a polynomial at any $\alpha \in F$ defines a linear functional

$$\begin{aligned} ev_\alpha: F[x] &\longrightarrow F \\ p(x) = \sum_{i=0}^{n} a_i x^i &\longmapsto \sum_{i=0}^{n} a_i \alpha^i =: p(\alpha) \qquad \alpha^0 := 1 \end{aligned}$$

(c) For any compact interval $[a,b] \subseteq \mathbb{R}$, integration defines a linear functional

$$\begin{aligned} \left\{ \begin{matrix} \text{continuous fnc} \\ [a,b] \longrightarrow \mathbb{R} \end{matrix} \right\} = C([a,b]) &\longrightarrow \mathbb{R} \\ f &\longmapsto \int_a^b f \, dx \end{aligned}$$

$\square$

3.1.3 Definition The dual space V^* of a vector space V is the set of linear functionals on V:

$$V^* := \mathrm{Hom}_F(V, F) \qquad \boxed{\begin{array}{l}(f+g)(v) := f(v)+g(v)\\ (\alpha f)(v) := \alpha f(v)\end{array}}$$

By Theorem 2.5.1, it is a vector space over F.

$\square$

Remark: Using Cor 2.6.3, we already know that if V finite-dimensional, then

$$\begin{aligned}\dim_F V^* &= \dim_F \mathrm{Hom}(V, F)\\ &= \dim_F V \cdot \dim_F F\\ &= \dim_F V.\end{aligned}$$

So if V is finite-dimensional, then $V \cong V^*$. So the first step to understanding V^* is to find an isomorphism.

3.1.4 Theorem (Dual Basis) Let V be a finite-dimensional vector space over F w/ basis $B = (b_1, \dots, b_n)$. There exists a unique basis $B^* = (\varphi_1, \dots, \varphi_n)$ for V^* such that

$$\varphi_i(b_j) = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases} =: \delta_{ij} \quad \text{"Kronecker Delta"}$$

for all $i, j \in \{1, \dots, n\}$. The basis B^* is called the dual basis to B. Moreover, if $f \in V^*$, then

$$f = \sum_{i=1}^{n} f(b_i)\varphi_i \qquad \left(\Leftrightarrow [f]_{B^*} = (f(b_1), \dots, f(b_n)) \right)$$

and if $v \in V$, then

$$v = \sum_{i=1}^{n} \varphi_i(v)\, b_i \qquad \left(\Leftrightarrow [v]_B = (\varphi_1(v), \dots, \varphi_n(v)) \right)$$

Proof.

For each $i = 1, \dots, n$ define $\varphi_i : V \longrightarrow F$ to be the unique linear map satisfying

$$\begin{tikzcd} b_j \arrow[r, mapsto] & \delta_{ij} & \\ B \arrow[r] \arrow[d] & F & \\ V \arrow[ur, dashed, "G", "\exists!"'] & & \text{Thm } 2.3.1 \end{tikzcd}$$

$$\varphi_i(b_j) = \delta_{ij}$$

for all $j = 1, \dots, n$. By Theorem 2.3.1, the φ_i are linear maps determined uniquely by their values on the basis B. So it suffices to show that $B^* = (\varphi_1, \dots, \varphi_n)$ is a basis for V^*. Then the uniqueness of the basis is immediate.

(B^* is independent) Suppose $\sum_{i=1}^{n} \alpha_i \varphi_i = 0_{V^*}$ (in V^*)

Then for each $j \in \{1, \dots, n\}$,

$$\begin{aligned} 0_F = 0_{V^*}(b_j) &= \left(\sum_{i=1}^{n} \alpha_i \varphi_i \right)(b_j) \\ &= \sum_{i=1}^{n} \alpha_i \varphi_i(b_j) \\ &= \sum_{i=1}^{n} \alpha_i \delta_{ij} \\ &= \alpha_j \delta_{jj} = \alpha_j . \end{aligned}$$

Thus, B^* is independent.

(B^* spans V^*) By dimension argument, B^* spans. But we have to prove the statements about coordinate vectors, which will follow from this part. Let $f \in V^*$. The claim is that

$$f = \sum_{i=1}^{n} f(b_i)\beta_i .$$

linear maps.

Both sides of the eq are linear maps, so it suffices to show that they have the same values on B. Let $j \in \{1, \dots, n\}$. Then

$$\begin{aligned}\left(\sum_{i=1}^{n} f(b_i)\beta_i\right)(b_j) &= \sum_{i=1}^{n} f(b_i)\beta_i(b_j) \\ &= \sum_{i=1}^{n} f(b_i)\delta_{ij} \\ &= f(b_j).\end{aligned}$$

Finally, let $v \in V$. The claim is that $v = \sum_{i=1}^{n} \beta_i(v) b_i$.

Suppose $[v]_B = (\alpha_1, \dots, \alpha_n)$. I claim $\alpha_j = \beta_j(v)$. We have

$$\begin{aligned}\beta_j(v) = \beta_j\left(\sum_{i=1}^{n} \alpha_i b_i\right) &= \sum_{i=1}^{n} \alpha_i \beta_j(b_i) \\ &= \alpha_j .\end{aligned}$$

This completes the proof.

$$\begin{aligned} b_i &\longmapsto \beta_i \\ B &\longrightarrow V^* \end{aligned}$$

$\square$

Remark: By the theorem, every basis B for V gives rise to a dual basis B^* for V^*. Sending B to B^* establishes an isomorphism $V \xrightarrow{\sim} V^*$. The linear functional f_i in B^* just gives the ith coordinate of $v \in V$ in the original basis B.

Question: Does every basis for V^* arise as the dual basis for some basis for V? We can answer this by considering the dual space $(V^*)^*$ of V^*.

3.1.5 Definition

Let V be a vector space over F. For each $v \in V$, define $ev_v \in V^{**}$ as follows:

$$\begin{aligned} ev_v : V^* &\longrightarrow F \\ f &\longmapsto f(v) \end{aligned}$$

That is, ev_v is the function which evaluates a linear functional $V \to F$ at $v \in V$. ☐

3.1.6 Theorem

If V is a finite-dimensional vector space, then there is an isomorphism

$$\begin{aligned} V &\longrightarrow V^{**} \\ v &\longmapsto ev_v \end{aligned}$$

of vector spaces.

Proof. First, one verifies that $ev_v \in V^{**}$. Exercise.

Then one checks that $\begin{aligned} V &\longrightarrow V^{**} \\ v &\longmapsto ev_v \end{aligned}$ is linear.

Exercise, Then one checks that the map is an isomorphism. We already know that $\dim V = \dim V^* = \dim V^{**}$ so it suffices to show that $V \longrightarrow V^{**}$ is injective, i.e. $\operatorname{Ker}(v \longmapsto ev_v) = \{0\}$. Suppose $0 \neq v \in V$. I claim that $ev_v \neq 0$. Since $v \neq 0$, $\{v\}$ is independent. Extend $\{v\}$ to a basis for V. Define $f_v : V \longrightarrow F$ by specifying its values on the basis:

$$f_v(v) = 1$$
$$f_v(w) = 0 \qquad (\text{for any basis vector } w \neq v).$$

Then $ev_v(f_v) = f_v(v) = 1$. Thus, $ev_v \neq 0$. By contrapositive, $ev_v = 0$ implies $v = 0$. Thus, $\ker(v \mapsto ev_v) = \{0\}$. This proves the claim.

$\square$

Remark: Under the identification $v \longmapsto ev_v$, one can think of V as being the dual space of V^*.

3.1.7 Corollary Let V be a finite-dimensional vector space. If $L \in V^{**}$, then there is a unique $v \in V$ such that

$$L(f) = f(v) \qquad (\forall f \in V^*)$$

In other words, linear functionals $V^* \longrightarrow F$ are just "evaluation" at different vectors in V.

Proof. Let $L \in V^{**}$. By Thm 3.1.6, there is a unique $v \in V$ such that $L = ev_v$. Thus,

$$L(f) = ev_v(f) = f(v).$$ □

3.1.8 Corollary Let V be a finite-dimensional vector space. Then every basis for V^* is the dual basis of a basis for V.

Proof. Let $C = (f_1, \dots, f_n)$ be a basis for V^*. Then $C^* = (g_1, \dots, g_n)$ is a basis V^{**} satisfying

$$g_i(f_j) = \delta_{ij} \qquad (\text{Thm } 3.1.4).$$

By Cor. 3.1.7, there are unique vectors $b_1, \dots, b_n \in V$ such that $g_j(f) = f(b_j)$ for all $f \in V^*$. In particular,

$$f_i(b_j) = g_j(f_i) = \delta_{ji}$$

So if $B = (b_1, \dots, b_n)$ is a basis for V then $C = B^*$ by uniqueness in Thm 3.1.4. But B is a basis for V because $b_1, \dots, b_n$ are the images of the elements $g_1, \dots, g_n$ of the basis C^* for V^{**} under the isomorphism $V^{**} \xrightarrow{\sim} V$. This proves the claim. □

Remark: V, V^*, V^{**} are all isomorphic because they all have the same dimension. Yet there is a sense in which V and V^{**} are more that just isomorphic – they are "naturally isomorphic", in contrast V and V^* are not naturally isomorphic.

Here's the idea: we obtain an isomorphism $V \xrightarrow{\sim} V^*$ by choosing a basis for V. There is no canonical basis. So there is no canonical isomorphism, every isomorphism involves an arbitrary choice. OTOH the isomorphism $V \xrightarrow{\sim} V^{**}$ was defined w/out specifying a basis at all. So, it is much more "natural".

The "unnaturallity" of $V \xrightarrow{\sim} V^*$ has the following consequence: there is no choice of a family of isomorphisms $\Phi_V : V \to V^*$, one for each f.d vector space V, that are compatible w/ linear maps $L : V \to W$ in the sense that the following diagram commutes:

$$\begin{tikzcd}
V \arrow[r, "\Phi_V"] \arrow[d, "L"'] \arrow[dr, phantom, "G"] & V^* \\
W \arrow[r, "\Phi_W"'] & W^* \arrow[u, "L^*"']
\end{tikzcd}$$

$$\boxed{\begin{array}{l} L^*(f) = f \circ L \\ \quad \text{for } f \in W^* \end{array}}$$

Any construction of Φ_V involves a choice of basis for each V, and since L is arbitrary, there is no reason why the diagram would commute. For instance, the diagram will never commute if $V \neq 0$ and $L : V \to W$ is the zero function.

OTOH: the family of isomorphisms $V \xrightarrow{\sim} V^{**}$ is natural!

3.1.9 Proposition Let $L: V \longrightarrow W$ be a linear map of finite-dimensional vector spaces. There is a linear map

$$\begin{aligned} L^{**}: V^{**} &\longrightarrow W^{**} \\ (f: V^* \to F) &\longmapsto f \circ L^* \end{aligned}$$

$$\boxed{\begin{aligned} L^*: W^* &\longrightarrow V^* \\ g &\longmapsto g \circ L \end{aligned}}$$

For any finite-dimensional vector spaces V and W and any linear map $L: V \longrightarrow W$, the following diagram commutes:

$$\begin{tikzcd}
v \arrow[r, mapsto] & ev_v \\
V \arrow[r] \arrow[d, "L"'] \arrow[dr, phantom, "\circlearrowright"] & V^{**} \arrow[d, "L^{**}"] \\
W \arrow[r] & W^{**} \\
w \arrow[r, mapsto] & ev_w
\end{tikzcd}$$

Proof.

Easy to check L^{**} is a linear map. Let $v \in V$. The claim is that

$$L^{**}(ev_v) = ev_{L(v)}.$$

Both sides are elements of W^{**}. Let $g \in W^*$.

Then
$$\begin{aligned} (L^{**}(ev_v))(g) &= (ev_v \circ L^*)(g) \\ &= ev_v(g \circ L) \\ &= (g \circ L)(v). \end{aligned}$$

OTOH:

$$ev_{L(v)}(g) = g(L(v)) = (g \circ L)(v).$$

Thus, the diagram commutes.

$\square$

Remark: In the 1940's – 1950's, a mathematician Saunders MacLane wanted to understand this notion of "naturality".

To precisely define

"natural isomorphism"

(ex: the family of iso $V \longrightarrow V^{**}$, $v \longmapsto ev_v$)

he had to define

"functor"

$$\left(\text{ex: } \begin{tikzcd} V \arrow[dd, "L"'] & & V^{**} \arrow[dd, "L^{**}"] \\ & \longmapsto & \\ W & & W^{**} \end{tikzcd} \right)$$

and to define functor he had to define

"category"

(ex: f.d vector spaces (objects)
linear maps (morphisms))

The branch of mathematics called "category theory" was born. Loosely speaking, categories are mathematical structures consisting of objects, arrows between objects, together with a way to compose arrows. Functors are the structure preserving maps between categories: they send objects to objects and arrows to arrows in a way that preserves composition. The dual and the double dual provide examples of functors because we have a way to take duals of linear maps. Natural isomorphism provides a notion of when two functors are essentially the same. ☐

3.2 The Dual of a Linear Map

We will study the construction of the dual of a linear map as seen above.

3.2.1 Definition Let $L: V \longrightarrow W$ be a linear map. The linear map

$$L^*: W^* \longrightarrow V^*$$
$$g \longmapsto g \circ L$$

is called the **dual** of L. Other names: transpose or adjoint. Exercise to check that it is a linear map. ☐

Note: L^{**} from 3.1.9 is just $(L^*)^*$

3.2.2 Theorem Let V and W be finite-dim. vector spaces with bases $B=(b_1, \dots, b_n)$ and $C=(c_1, \dots, c_m)$. Let $B^* = (\beta_1, \dots, \beta_n)$ and $C^* = (\psi_1, \dots, \psi_m)$ be the dual bases for V^* and W^*. Let $L: V \longrightarrow W$ be a linear map. Then

$$[L^*]^{B^*}_{C^*} = \left([L]^C_B\right)^T$$

Proof. The matrix $[L^*]^{B^*}_{C^*}$ is the unique matrix w/ the property that

$$[L^*]^{B^*}_{C^*} [g]_{C^*} = [L^*(g)]_{B^*}$$

for all $g \in W^*$. I will show that $([L]_B^C)^T$ has the same property. Let $g \in W^*$. Then

Dual Basis Thm

$$[g]_{C^*} = (g(c_1), \dots, g(c_m))$$

$$[L^*(g)]_{B^*} = [g \circ L]_{B^*} = (g\circ L(b_1), \dots, g \circ L(b_n))$$

$$\forall i : [L(b_i)]_C = (\psi_1(L(b_i)), \dots, \psi_m(L(b_i)))$$

$$([L]_B^C)^T [g]_{C^*} = \begin{pmatrix} [L(b_1)]_C \\ \vdots \\ [L(b_n)]_C \end{pmatrix} \begin{pmatrix} g(c_1) \\ \vdots \\ g(c_m) \end{pmatrix}$$

$$g = \sum_{j=1}^m g(c_j)\psi_j$$

$$= \begin{pmatrix} \psi_1(L(b_1)) & \cdots & \psi_m(L(b_1)) \\ & \vdots & \\ \psi_1(L(b_n)) & \cdots & \psi_m(L(b_n)) \end{pmatrix} \begin{pmatrix} g(c_1) \\ \vdots \\ g(c_m) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^m \psi_j(L(b_1))\, g(c_j) \\ \vdots \\ \sum_{j=1}^m \psi_j(L(b_n))\, g(c_j) \end{pmatrix}$$

$$= \begin{pmatrix} g(L(b_1)) \\ \vdots \\ g(L(b_n)) \end{pmatrix} = [L^*(g)]_{B^*}. \quad \square$$

3.2.3 Proposition

Let $L: V \longrightarrow W$ be a linear map w/ V and W finite-dimensional. Then

$$\operatorname{rank} L = \dim(\operatorname{im} L) = \dim(\operatorname{im} L^*) = \operatorname{rank} L^*$$

Proof By the above, $[L^*]_{C^*}^{B^*} = \left([L]_B^C\right)^T$. So they have the same rank. But

$$\operatorname{rank}[L]_B^C = \operatorname{rank}\left([L]_B^C\right)^T$$

because the column space of $\left([L]_B^C\right)^T$ is just the row space of $[L]_B^C$. In Section 2.4, I proved that row and column space have the same dimension. $\square$

3.3 Multilinear Maps

Motivation: In your last linear algebra class, you probably studied the determinant of a square matrix. It defines a function

$$\det : F^{n\times n} \longrightarrow F$$

It turns out to be useful. You probably know several ways to compute the determinant (cofactor expansion, explicit formulas for small n), but not a formal definition. Writing down an explicit formula is somewhat involved and not necessarily enlightening.

A different approach is to think about the properties the determinant has and try to charactize it as the unique function

$$F^{n\times n} \longrightarrow F$$

w/ those properties. It is useful to think of the determinant as a function on the columns of the matrices

$$F^{n\times n} = F^n \times \cdots \times F^n$$

$$\det : F^n \times \cdots \times F^n \longrightarrow F$$
$$(c_1, \ldots, c_n) \longmapsto \det(c_1, \ldots, c_n)$$

From this viewpoint, the determinant is

(a) "multilinear" i.e. linear when restricted to a single column

(b) "alternating" i.e. $c_i = c_j$, $i \neq j \Rightarrow \det(c_1, \ldots, c_n) = 0$

It turns out that the space of all alternating multilinear map

$$F^n \times \cdots \times F^n \longrightarrow F$$

is one-dimensional. There is a unique alternating multilinear map sending $I_n \mapsto 1$. This function is the determinant.