2.8.5 Proposition Let $V_1, \dots, V_n$ be subspaces of V. Then $V = \bigoplus_{i=1}^{n} V_i$ if and only if there are bases $B_1, \dots, B_n$ for $V_1, \dots, V_n$ such that $B = \bigcup_{i=1}^{n} B_i$ is a basis for V.

Proof. Exercise. Proof by induction. $\square$

2.8.6 Corollary If $V_1, \dots, V_n$ are vector spaces over F, then

$$\dim_F\left(\bigoplus_{i=1}^{n} V_i\right) = \sum_{i=1}^{n} \dim_F(V_i).$$

$\square$

2.8.7 Definition Let W be a subspace of V. A **complement** of W is a subspace U of V such that $V = W \oplus U$. $\square$

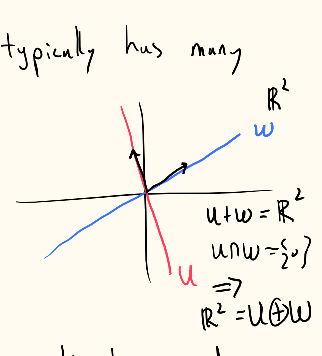
Remark: Every subspace has a complement. Let $\{w_1, \dots, w_n\}$ be a basis for $W \subseteq V$. Extend to a basis $\{w_1, \dots, w_n, w_{n+1}, \dots, w_m\}$ for V. Then

$$U := \operatorname{span}\{w_{n+1}, \dots, w_m\}$$

is a complement of W. If V is infinite dimensional, the statement can be proved using AoC. $\square$

2.8.8 Example A subspace typically has many complements.

- $V = \mathbb{R}^2$, W = any line thru origin



A vector space typically has many direct sum decompositions. If $B = \{b_1, \dots, b_n\}$ is a basis for V, then

$$V = \operatorname{Span}(b_1) \oplus \cdots \oplus \operatorname{Span}(b_n).$$

$\square$

2.8.9 Definition Let V be a vector space and let W be a subspace. We define a relation $\sim_W$ on V as follows:

$$u \sim_W v \text{ if and only if } u - v \in W.$$

$\square$

2.8.10 Proposition Let V be a vector space and let W be a subspace. Then

(a) $\sim_W$ is an equivalence relation

(b) The equivalence class of $v \in V$ is the set

$$v + W := \{v + w \mid w \in W\}$$

Proof. (a) (reflexive) Let $x \in V$. Then $x - x = 0 \in W$. Thus, $x \sim_W x$.

(symmetric) Assume $x \sim_W y$. Then $x - y \in W$. Hence, $y - x = (-1)(x - y) \in W$. So $y \sim_W x$

(transitive) Assume $x \sim_W y$ and $y \sim_W z$. Then $x - y \in W$ and $y - z \in W$. Hence, $x - z = x - y + y - z \in W$. Thus, $x \sim_W z$.

(b) Write $[v]$ for the equivalence class of $v \in V$ w.r.t $\sim_W$. Then

$$\begin{aligned} [v] &= \{u \in V \mid u \sim_W v\} \\ &= \{u \in V \mid u - v \in W\} \\ &= \{v + w \mid w \in W\} = v + W. \end{aligned}$$

$\square$

2.8.11 Definition Let V be a vector space and W a subspace. The equivalence classes $v + W$ are called **cosets of V w.r.t. W**. The **quotient of V by W** is the set V/W of all cosets

$$V/W := \{v + W \mid v \in V\}.$$

together w/ the operations of addition and scalar multiplication defined as follows:

(addition) $(u + W) + (v + W) := (u + v) + W$

(scalar mult) $\alpha(u + W) := (\alpha u) + W$.

$\square$

2.8.12 Proposition The operations of addition and scalar multiplication on V/W are well-defined, making V/W into a vector space.

Proof. We must show that the operations are well-defined, i.e., they do not depend on the choice of representatives for the cosets involved.

Suppose $u + W = u' + W$ and $v + W = v' + W$.

$u - u' \in W$ $v - v' \in W$

Fact: $w \in W$ iff $w + W = W$

We have

$$\begin{aligned} (u + W) + (v + W) &= (u + v) + W \\ &= (u + v) + \underbrace{(u' - u)}_{\in W} + \underbrace{(v' - v)}_{\in W} + W \\ &= (u' + v') + W \\ &= (u' + W) + (v' + W). \end{aligned}$$

Let $\alpha \in F$. Then

$$\begin{aligned} \alpha(u + W) &= \alpha u + W \\ &= \alpha u + \underbrace{(\alpha u' - \alpha u)}_{\in W} + W \\ &= \alpha u' + W \\ &= \alpha(u' + W). \end{aligned}$$

Easy to check that V/W is an F-vector space w/ these operations. The zero vector is $W = 0 + W$. Additive inverse of $v + W$ is $-(v + W) := -v + W$. $\square$

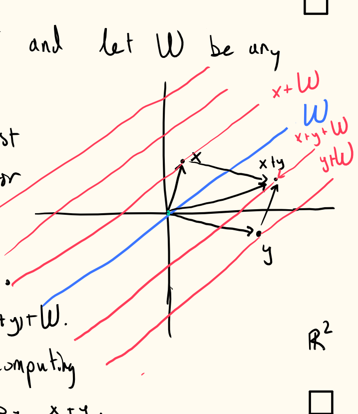
2.8.13 Example Let $V = \mathbb{R}^2$ and let W be any line through the origin.

Let $x \in \mathbb{R}^2$. Then $x + W$ is just a translation of W by the vector x. So

$V/W = \{\text{all lines parallel to } W\}$.

By def $x + W + y + W = (x + y) + W$.

So two lines are added by computing $x + y$ and then translating W by $x + y$.



$\square$

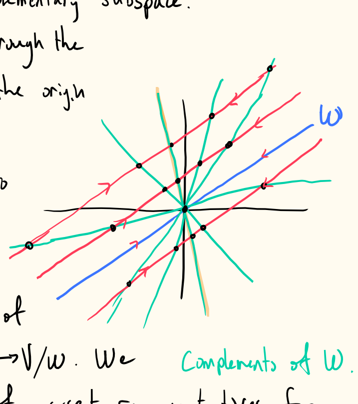
Remark: The quotient space V/W comes equipped w/ a special linear map

$$\begin{aligned} \pi : V &\longrightarrow V/W \\ v &\longmapsto v + W \end{aligned}$$

called the projection map or quotient map. Notice: π is surjective and $\ker \pi = W$. In the above example, the map $\pi : \mathbb{R}^2 \to \mathbb{R}^2/W$ collapses each line parallel to W to a single point.

Some Motivation: Given a subspace V of W, there is no canonical choice of complementary subspace. For example, $V = \mathbb{R}^2$, W = line through the origin. Any nonparallel line through the origin is a complement of W.

Observe: any line (coset) parallel to W intersects every complement of W at a unique point. In other words, given any complement U of W, we have a bijection $U \leftrightarrow V/W$. We can think of U as a set of coset representatives for V/W (U contains precisely one element from each coset).



2.8.14 Proposition Let U be any complement of W in V. The composition of linear maps

$$U \xhookrightarrow{i} V \xrightarrow{\pi} V/W$$

is an isomorphism of U onto V/W. In particular,

$$V \cong W \oplus V/W.$$

Proof. Clearly, $\pi \circ i$ is linear because π and i are. By assumption, $V = U \oplus W$. We need to show that $\pi \circ i$ is bijective.

(surjective) Let $v \in V$. Then there exists unique elements $u \in U$ and $w \in W$ such that $v = u + w$. Then

$$\begin{aligned} \pi \circ i(u) = \pi(u) = u + W &= u + w + W \\ &= v + W. \end{aligned}$$

So $\pi \circ i$ is surjective.

(injective) Suppose $u + W = \pi \circ i(u) = \pi \circ i(u') = u' + W$. Then $u - u' \in W \cap U = \{0\}$. Hence $u = u'$. So $\pi \circ i$ is injective. Note: injectivity of $\pi \circ i$ says precisely that U is a set of representatives for V/W.

Finally, $V = W \oplus U \cong W \oplus V/W$. $\square$

Remark: By Prop 2.8.14, one can think of V/W as a "canonical" or "universal" complement of W, although V/W is NOT a subspace of V at all.

2.8.5 Theorem (Universal Property of the Quotient) Let W be a subspace of a vector space V. For every vector space Z and every linear map $L : V \to Z$ satisfying $W \subseteq \ker L$, there exists a unique linear map $\tilde{L} : V/W \to Z$ such that $\tilde{L} \circ \pi = L$. In other words, there is a unique linear map $\tilde{L}$ making the following

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\pi"'] \arrow[dr, phantom, "G"] & Z \\ V/W \arrow[ur, dashed, "\exists! \tilde{L}"'] & \end{tikzcd}$$

diagram.

Proof. Define $\tilde{L} : V/W \to Z$ via $\tilde{L}(v + W) = L(v)$. Then clearly $\tilde{L} \circ \pi = L$, as desired. The question is whether this is well-defined.

(well-defined) Assume $v + W = v' + W$. Then $v - v' \in W$. But $W \subseteq \ker(L)$ implies $L(v) - L(v') = L(v - v') = 0$. Hence,

$$\tilde{L}(v + W) = L(v) = L(v') = \tilde{L}(v' + W).$$

(linearity) We have

$$\begin{aligned} \tilde{L}(v + W + v' + W) &= \tilde{L}((v + v') + W) \\ &= L(v + v') \\ &= L(v) + L(v') \\ &= \tilde{L}(v + W) + \tilde{L}(v' + W). \end{aligned}$$

Similarly,

$$\begin{aligned} \tilde{L}(\alpha(v + W)) &= \tilde{L}(\alpha v + W) \\ &= L(\alpha v) \\ &= \alpha L(v) \\ &= \alpha \tilde{L}(v + W). \end{aligned}$$

So $\tilde{L}$ is a linear map satisfying $\tilde{L} \circ \pi = L$. This proves existence.

(uniqueness) Suppose $K : V/W \to Z$ is another linear map satisfying $K \circ \pi = L$. Then for any $v \in V$,

$$K(v + W) = K(\pi(v)) = K \circ \pi(v) = L(v) = \tilde{L}(v + W).$$

This proves that $\tilde{L}$ is unique w/ the property that $\tilde{L} \circ \pi = L$. $\square$

Remark: The quotient map $\pi : V \to V/W$ is a linear map w/ the following property:

$$W \subseteq \ker \pi \qquad (\pi \text{ "annihilates" } W)$$

The theorem states that π is the "universal" linear map defined on V which annihilates W. Any other linear map L which annihilates W "factors uniquely through π": $\tilde{L} \circ \pi = L$.

Using the language of category theory, one can give a precise definition of Universal Property. $\square$

Remark: The best way to think about the Universal Property of the Quotient Space is as an answer to the following question:

Question: How do I define linear maps out of the quotient space V/W?

Answer: Define a linear map on V whose kernel contains W.

In other words, there is a bijection

$$\begin{aligned} \operatorname{Hom}_F(V/W, Z) &\xleftrightarrow{\sim} \{L \in \operatorname{Hom}(V, Z) \mid W \subseteq \ker L\} \\ K &\longmapsto K \circ \pi \\ \tilde{L} &\longleftarrow\!\shortmid L \end{aligned}$$

where Z is any other vector space. Actually, the LHS is a vector space. Q: Is the RHS a subspace of $\operatorname{Hom}(V, Z)$? If yes, is the bijection an isomorphism of vector spaces?

2.8.16 Corollary (First Isomorphism Theorem) Let $L : V \to W$ be a linear map. Then there is a canonical isomorphism

$$V/\ker(L) \xrightarrow{\sim} \operatorname{im}(L)$$

More precisely, there is a unique isomorphism making the following diagram

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\pi"'] \arrow[dr, phantom, "G"] & W \\ V/\ker(L) \arrow[r, dashed, "\exists! \sim"'] & \operatorname{im}(L) \arrow[u, hook, "\iota"'] \end{tikzcd}$$

commute.

Proof. Clearly, L annihilates $\ker(L)$. So by UPoQ, there is a unique linear map $\tilde{L}$ making the following diagram commute

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\pi"'] \arrow[dr, phantom, "G"] & W \\ V/\ker(L) \arrow[ur, dashed, "\exists! \tilde{L}"'] & \end{tikzcd}$$

One can prove that $\operatorname{im}(L) = \operatorname{im}(\tilde{L})$ so we get a commutative diagram

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\pi"'] \arrow[dr, phantom, "G"] & W \\ V/\ker L \arrow[r, two heads, "\tilde{L}"'] & \operatorname{im}(L) = \operatorname{im}(\tilde{L}) \arrow[u, hook, "\iota"'] \end{tikzcd}$$

So we just need to show $\tilde{L}$ is injective. But one can prove that $\ker \tilde{L} = \ker L / \ker L = 0$. Thus, $\tilde{L}$ is injective. $\square$

Remark: Thus, every linear map factors through a quotient map, an isomorphism of vector space, and an inclusion map. This suggests quotient spaces are important.

2.8.17 Theorem (Rank-Nullity) Let V be a finite-dimensional vector space and let $L : V \to W$ be a linear map. Then

$$\dim(V) = \operatorname{rank}(L) + \operatorname{nullity}(L).$$

Proof. Since $\ker L$ is a subspace of V, we have an isomorphism

$$V \cong \ker(L) \oplus V/\ker(L).$$

By the First Isomorphism Theorem, $V/\ker(L) \cong \operatorname{im}(L)$. Thus,

$$\begin{aligned} \dim(V) &= \dim(\ker(L) \oplus V/\ker(L)) \\ &= \dim(\ker(L)) + \dim(\operatorname{im}(L)) \\ &= \operatorname{nullity}(L) + \operatorname{rank}(L). \end{aligned}$$

$\square$

Remark: One does not need quotient spaces to prove Rank-Nullity. Instead, one can start w/ a basis for $\ker(L)$, extend to a basis for V, and then check that the image of this basis is a basis for the image.