Remark: The choice of an ordered basis $B=(b_1,\dots,b_n)$ and a list of vectors $c_1,\dots,c_n \in W$ amounts to the choice of a function

$$f: B \longrightarrow W \qquad \left(\begin{array}{c}\text{namely}\\ f(b_i)=c_i\end{array}\right)$$

There is another function $i: B \overset{\subseteq}{\longrightarrow} V$, $b \longmapsto b$. The theorem can be restated as follows: for any function $f: B \to W$, there exists a unique linear map $L: V \to W$ such that $L \circ i = f$, i.e., the diagram

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "i"'] & W \\ V \arrow[ur, dashed, "\exists ! L"'] & \end{tikzcd} \qquad L\Big(\sum_{b\in B}\alpha_b b\Big) = \sum_{b\in B}\alpha_b L(b)$$

commutes.

Equivalently, the theorem establishes a bijection of sets

$$\left\{\begin{array}{c}\text{linear maps}\\ L: V \to W\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{functions}\\ f: B \to W\end{array}\right\} = W^B$$

$$L \longmapsto L\circ i$$
$$L \longleftarrow f$$

□

2.3.2 Proposition Let V be a finite-dimensional vector space w/ basis $B=\{b_1,\dots,b_n\}$ and let $L: V \to W$ be a linear map. Then

(a) L is injective iff $L(B)=\{L(b_1),\dots,L(b_n)\}$ is independent in W

(b) L is surjective iff $L(B)$ spans W

(c) L is an isomorphism iff $L(B)$ is a basis for W.

Proof. Clearly, (c) follows from (a) and (b).

(a) For any $\alpha_1,\dots,\alpha_n$ we have

$$\sum_i \alpha_i L(b_i) = L\Big(\sum_i \alpha_i b_i\Big).$$

If L is injective and LHS $=0$, then RHS $=0$. Hence, $\sum \alpha_i b_i \in \ker L = \{0\}$. So $\sum \alpha_i b_i = 0$. But B is independent, so $\alpha_1=\alpha_2=\dots=\alpha_n=0$. OTOH if $L(B)$ is independent and RHS $=0$, then LHS $=0$. By independence, $\alpha_1=\dots=\alpha_n=0$. So $\sum_i \alpha_i b_i=0$. This implies that $\ker L = \{0\}$ so that L is injective.

(b) ($\Rightarrow$) Let $w \in W$. Choose $v = \sum \alpha_i b_i$ s.t. $L(v)=w$. Then

$$w = L(v) = \sum \alpha_i L(b_i) \in \operatorname{Span}(L(B)).$$

Hence, $W = \operatorname{span}(L(B))$.

($\Leftarrow$) Let $w \in W$. Since $L(B)$ spans W, we can write

$$w = \sum_i \alpha_i L(b_i) = L\Big(\sum_i \alpha_i b_i\Big).$$

Set $v = \sum \alpha_i b_i$. Then $L(v)=w$. So L is surj. □

Remark: one can also prove that if $S=\{s_1,\dots,s_n\}$ spans V, then $L(S)$ spans $\operatorname{im}(L)$.

□

2.3.3 Corollary Suppose V and W are finite-dimensional vector spaces w/ $\dim V = \dim W$. Let $L: V \to W$ be a linear map. The following are equivalent:

(a) L is injective

(b) L is surjective

(c) L is an isomorphism.

Proof. Clearly, (a) and (b) $\Leftrightarrow$ (c). Let B be a basis for V. If L is injective, then $L(B)$ is independent by Prop 2.3.2(a). So $|L(B)| = |B| = \dim V = \dim W$. By Prop 1.4.15(a), $L(B)$ is a basis for W. By Prop 2.3.2(b), L is surjective. If L is surjective, then $L(B)$ spans W and has cardinality equal to $\dim W$. So $L(B)$ is a basis by 1.4.15(b). Hence, L is injective by Prop 2.3.2(a).

□

2.3.4 Theorem Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.

Proof. Let V and W be f.d. vector spaces.

($\Rightarrow$) If $L: V \to W$ is an isomorphism and B is a basis for V, then $L(B)$ is a basis for W. Hence

$$\dim V = |B| = |L(B)| = \dim W.$$

($\Leftarrow$) Suppose $\dim V = \dim W$. Then $V \cong F^{\dim V}$ and $W \cong F^{\dim W}$. Hence $V \cong W$ since $F^{\dim V} = F^{\dim W}$.

□

Remark: • Finite dimensional vector spaces are classified up to isomorphism by their dimension.

• Each is isomorphic to F^n for some $n \in \mathbb{N}$.

So, why do we bother studying abstract f.d. vector spaces at all? Why not just study F^n? The reason is that choosing an isomorphism $V \xrightarrow{\sim} F^n$ amounts to choosing an ordered basis for V. If $B=(b_1,\dots,b_n)$ is a basis for V, then an iso is given by

$$\begin{array}{ccc} V & \xrightarrow{\sim} & F^n \\ v & \longmapsto & [v]_B \\ b_i & \longleftarrow & e_i \end{array}$$

One can think of this isomorphism as assigning a coordinate system to V. However, a different choice of basis will result in a different coordinate system. But given an arbitrary vector space V, there is no natural/canonical/standard choice of basis.

□

2.4 Change of Basis

Recall: An $n\times n$ matrix $A \in F^{n\times n}$ is invertible if there exists a (unique) matrix $A^{-1} \in F^{n\times n}$ such that

$$AA^{-1} = I_n = A^{-1}A$$

where $I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1\end{pmatrix}$.

Recall: Let F be field. In Prop 2.1.2, we proved that there is a bijection

$$\begin{array}{ccccc} \operatorname{End}(F^n) := \left\{\begin{array}{c}\text{linear maps}\\ F^n \to F^n\end{array}\right\} & \overset{\sim}{\longleftrightarrow} & \left\{\begin{array}{c} n\times n \text{ matrices}\\ \text{over } F\end{array}\right\} = F^{n\times n} \\ \cup\!\shortmid & & \cup\!\shortmid \\ \left\{\begin{array}{c}\text{isomorphisms}\\ F^n \to F^n\end{array}\right\} & \overset{\sim}{\longleftrightarrow} & \left\{\begin{array}{c}\text{invertible}\\ \text{matrices}\end{array}\right\} \end{array}$$

The bijection was defined by sending $L: F^n \to F^n$ to the matrix $[L] = (L(e_1)|\cdots|L(e_n))$. The inverse sends $A \in F^{n\times n}$ to the linear map $A\cdot : F^n \to F^n$.

Observe:

$$\begin{array}{ccc} L: F^n \to F^n \text{ is an isomorphism} & \Leftrightarrow & [L] \text{ is invertible} \\ \Updownarrow & & \Updownarrow \\ L(e_1),\dots,L(e_n) \text{ basis for } F^n & \Leftrightarrow & \text{columns of } [L] \text{ basis for } F^n \\ \Updownarrow & & \Updownarrow \\ L(e_1),\dots,L(e_n) \text{ indep in } F^n & \Leftrightarrow & \text{columns of } [L] \text{ independent in } F^n \\ \Updownarrow & & \Updownarrow \\ L(e_1),\dots,L(e_n) \text{ span } F^n & \Leftrightarrow & \text{columns of } [L] \text{ span } F^n \end{array}$$

Thus, the bijection restricts to a bijection from isomorphisms to invertible matrices.

Recall: Choosing a basis $B=(b_1,\dots,b_n)$ for a vector space V over F determines an isomorphism

$$\begin{array}{ccc} V & \xrightarrow{\sim} & F^n \\ v & \longmapsto & [v]_B \end{array}$$

Suppose $C=(c_1,\dots,c_n)$ is another basis for V. Consider the composition

$$\begin{array}{ccccc} F^n & \xrightarrow{\sim} & V & \xrightarrow{\sim} & F^n \\ [v]_B & \longmapsto & v & \longmapsto & [v]_C \\ e_i & \longmapsto & b_i & \longmapsto & [b_i]_C \end{array}$$

This is an isomorphism $F^n \to F^n$. Therefore it is given by an invertible matrix:

2.4.1 Definition Let $B=(b_1,\dots,b_n)$ and $C=(c_1,\dots,c_n)$ be ordered bases for V. The transition matrix (or change of basis matrix) from B to C is the $n\times n$ matrix $T_B^C \in F^{n\times n}$ whose i^{th} column is $[b_i]_C$, $1\le i\le n$. I.e.

$$T_B^C = ([b_1]_C|\cdots|[b_n]_C).$$

□

2.4.2 Proposition Let $B=(b_1,\dots,b_n)$ and $C=(c_1,\dots,c_n)$ be ordered bases for V. Then

(a) $T_B^C \in F^{n\times n}$ is the unique $n\times n$ matrix satisfying $T_B^C[v]_B = [v]_C$ for all $v\in V$. In other words, T_B^C is the unique matrix making the diagram

$$\begin{tikzcd} & F^n \arrow[d, "T_B^C\cdot"] \\ V \arrow[ur, "\sim", "v \mapsto {[v]_B}"'] \arrow[dr, "\sim"', "v \mapsto {[v]_C}"] & \\ & F^n \end{tikzcd}$$

commute.

(b) T_B^C is invertible with $(T_B^C)^{-1} = T_C^B$

(c) (change of basis is transitive) If $D=(d_1,\dots,d_n)$ is a third basis for V, then $T_B^D = T_C^D T_B^C$.

Proof.

(a) Let $v\in V$. Then $T_B^C[v]_B = [v]_C$ by definition. Suppose $A\in F^{n\times n}$ satisfying $A[v]_B = [v]_C$ for all $v\in V$. Then

$$i^{th}\text{ column of } A = Ae_i = A[b_i]_B = [b_i]_C$$
$$= i^{th}\text{ column of } T_B^C.$$

(b) We already know T_B^C is invertible because it is the matrix representing the isomorphism $F^n \xrightarrow{\sim} F^n$, $[v]_B \longmapsto [v]_C$.

Let $v\in V$. Then using (a),

$$[v]_B = T_C^B[v]_C = T_C^B T_B^C [v]_B \Rightarrow T_C^B T_B^C = I_n$$
$$[v]_C = T_B^C[v]_B = T_B^C T_C^B [v]_C \Rightarrow T_B^C T_C^B = I_n$$

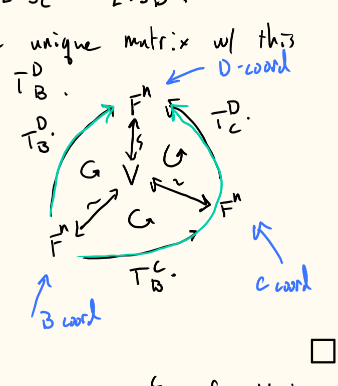
Thus, $(T_B^C)^{-1} = T_C^B$ by uniqueness.

(c) Let $v\in V$. Then

$$T_C^D T_B^C [v]_B = T_C^D[v]_C = [v]_D.$$

By part (a), T_B^D is the unique matrix w/ this property. Thus, $T_C^D T_B^C = T_B^D$.

Alternative proof: Meditate upon the diagram



□

Note: part (c) is useful for computing T_B^C if V has a standard basis E. If E is standard, then T_B^E and $(T_C^E)^{-1}$ are easy to compute and $T_B^C = (T_C^E)^{-1}T_B^E$.

2.4.3 Example $E=((1,0),(0,1))$, $B=((1,2),(3,4))$, $C=((-1,1),(1,2))$ are bases for $\mathbb{R}^2$. Compute T_B^C.

Use part (c). $(1,2) = \alpha(1,0)+\beta(0,1)$

$$T_B^E = ([(1,2)]_E \ [(3,4)]_E) = \begin{pmatrix} 1 & 3 \\ 2 & 4\end{pmatrix} \qquad (T_C^E)^{-1} = ([(-1,1)]_E \ [(1,2)]_E)^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 2\end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} -2 & 1 \\ 1 & 1\end{pmatrix}$$

Then $T_B^C = (T_C^E)^{-1}T_B^E = \frac{1}{3}\begin{pmatrix} -2 & 1 \\ 1 & 1\end{pmatrix}\cdot\begin{pmatrix} 1 & 3 \\ 2 & 4\end{pmatrix} = \begin{pmatrix} 0 & -2/3 \\ 1 & 7/3\end{pmatrix}$ □

2.5 Algebra of Linear Maps

Notation: Let V and W be vector spaces over F. The set of all linear maps $L: V\to W$ is denoted by

$$\operatorname{Hom}_F(V,W) \subseteq \operatorname{Maps}(V,W) = W^V.$$

One writes $\operatorname{End}_F(V) := \operatorname{Hom}_F(V,V)$.

Recall: Since W is a vector space, W^V is a vector space w/ the operations

$$(f+g)(v) := f(v)+g(v) \qquad \forall f,g \in W^V$$
$$(\alpha f)(v) := \alpha f(v) \qquad \alpha \in F.$$

2.5.1 Theorem Let V and W be vector spaces over F. Then the set $\operatorname{Hom}_F(V,W)$ of linear maps from V to W is a subspace of the vector space $W^V = \operatorname{Maps}(V,W)$ of all functions from V to W.

Proof. Clearly, $0: V \to W$ is a linear map. It is easy to check that the sum of two linear maps is linear. Similarly, the scalar multiple of a linear map is linear. □

Remark When $V=W$, we can define a binary product on $\operatorname{End}(V)$: for $L,K: V\to V$, the product of L and K is defined as follows:

$$KL := K\circ L.$$

We already proved that $KL \in \operatorname{End}(V)$ in Prop 2.1.10.

2.5.2 Proposition Let $\alpha \in F$ and $L,S,T \in \operatorname{End}_F(V)$. Then

(a) $L(ST) = (LS)T$

(b) $L(S+T) = LS+LT$

(c) $(S+T)L = SL+TL$

(d) $\alpha(ST) = (\alpha S)T = S(\alpha T)$.

Proof. Exercise.

□

Remark So $\operatorname{End}(V)$ is an F-vector space, but it also has a way to "multiply" vectors and this operation is compatible with addition and scalar multip. in $\operatorname{End}(V)$. Such an algebraic structure is called an algebra.

2.5.3 Example (algebras)

(a) $\operatorname{End}_F(V)$ over F

(b) F over F

(c) $F^{n\times n}$ over F (using matrix multiplication)

(d) If $K \subseteq F$ is a subfield, then F is a K-algebra. e.g. $\mathbb{C}$ over $\mathbb{R}$ or $\mathbb{Q}$.

(e) $F[x]$ over F (using ordinary multiplication of polynomials)

Note: $F_n[x]$ is not an algebra over F since multiplication increases degree. □