1.3.2 Example (subspaces) Let F be a field.

(a) A vector space V always has two subspaces: $\{0\}$, V.

(b) let $W \subseteq F^n$ be the set of solutions to the system of m linear equations over F:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{mn}x_1 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

then W is a subspace of F^n.

(c) Let $F_n[x]$ denote the set of polynomials in $F[x]$ w/ degree $\leq n$. This is a subspace of $F[x]$.

(d) $F^{n\times n} = \mathrm{Mat}_{n\times n}(F)$ has lots of interesting subspaces:

(i) diagonal matrices: $A = (a_{ij})$ w/ $a_{ij} = 0$ if $i \neq j$

(ii) symmetric matrices: $A = (a_{ij})$ w/ $a_{ij} = a_{ji}$.

(iii) skew-symmetric matrices: $A = (a_{ij})$ w/ $a_{ij} = -a_{ji}$.

(iv) upper triangular matrices: $A = (a_{ij})$ $a_{ij} = 0$ if $i > j$

(v) Magic squares: a square matrix is magic if all its rows, columns, main diagonal, off diagonal have same sum:

all the same "magic sum"

$$\begin{cases} \text{(rows)} & \sum_{j=1}^{n} a_{ij} \quad i = 1, \dots, n \\ \text{(columns)} & \sum_{i=1}^{n} a_{ij} \quad j = 1, \dots, n \\ \text{(main diag)} & \sum_{i=1}^{n} a_{ii} \\ \text{(off diag)} & \sum_{i=1}^{n} a_{n-(i-1),i} \end{cases}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \in Mag_3(\mathbb{Z}_3)$$

$$\begin{bmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{bmatrix} \in Mag_3(\mathbb{Q})$$

The set $Mag_n(F)$ of $n \times n$ magic squares is a subspace of $F^{n \times n}$.

(e) Subspaces of $\mathbb{R}^n$

(n=1) $\{0\}$, $\mathbb{R}$

(n=2) $\{0\}$, $\mathbb{R}^2$, lines through the origin

(n=3) $\{0\}$, $\mathbb{R}^3$, lines through the origin
planes through the origin. $\square$

1.3.3 Proposition (intersection of subspaces) Let V be a vector space over F. The intersection of any collection of subspaces of V is again a subspace of V.

Proof. Exercise. $\square$

1.3.4 Definition Let S be a subset of a vector space V over F. The **span** of S is the subspace

$$\operatorname{Span}(S) := \bigcap_{\substack{W \text{ a subspace of } V \\ S \subseteq W}} W .$$

By Prop 1.3.3, this is a subspace. By def, it is the smallest subspace of V which contains. Note: $\operatorname{span}(\phi) = \{0\}$.

$\square$

1.3.5 Definition Let $v_1, \dots, v_n$ be vectors in a vector space V over F. A **linear combination** of $v_1, \dots, v_n$ is a vector of the form

$$\sum_{i=1}^{n} \alpha_i v_i = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

for some $\alpha_1, \dots, \alpha_n \in F$. A linear combination is called **trivial** if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

$\square$

1.3.6 Proposition **(span)** Let S be a subset of a vector space V over F. Then $\operatorname{span}(S)$ is equal to the set of all possible finite linear combos of the vectors in S:

$$\operatorname{Span}(S) = \left\{ \sum_{i=1}^{n} \alpha_i v_i \;\middle|\; n \in \mathbb{N},\ \alpha_1, \dots, \alpha_n \in F,\ v_1, \dots, v_n \in S \right\}$$

Proof. ($\subseteq$) Denote the RHS by W. Suffices to show that W is a subspace containing S. Clearly it contains S. Clearly $0 \in W$ by taking the trivial linear combo. Also, the sum of two linear combos in S is also a linear combo in S. The scalar multiple of a linear combo in S is clearly a linear combo in S,

($\supseteq$) Clear because $S \subseteq \mathrm{Span}(S)$ and $\mathrm{span}(S)$ is a subspace. $\square$

Note: The set S could be infinite. But if $S = \{s_1, \dots, s_n\}$, then $\mathrm{Span}(S) = \left\{\sum_{i=1}^{n} \alpha_i s_i \mid \alpha_i \in F\right\}$.

1.3.7 Example (span) Consider $S = \{1, x\} \subseteq F[X]$. Then $\mathrm{span}(S) = F_1[X]$. More gen.,

$$\mathrm{span}\{1, x, x^2, \dots, x^n\} = F_n[x]. \qquad \square$$

1.3.8 Definition Let V be a vector space. Let Γ be any set and suppose that W_γ is a subspace of V for each $\gamma \in \Gamma$. Their sum is the subspace

$$\sum_{\gamma \in \Gamma} W_\gamma := \mathrm{Span}\left(\bigcup_{\gamma \in \Gamma} W_\gamma\right). \qquad \square$$

Note: The set Γ could be infinite. If $\Gamma = \{1, \dots, n\}$ is finite, then

$$\sum_{i=1}^{n} W_i = W_1 + \dots + W_n = \{w_1 + \dots + w_n \mid w_i \in W_i,\ i = 1, \dots, n\}.$$

This explains why its called the sum of subspaces!

1.3.9 Definition A subset S of a vector space V is called a spanning set for V if $\operatorname{Span}(S) = V$.

$\square$

1.3.10 Example (spanning sets)

(a) $S = \{(1,0), (0,1)\}$ spans $\mathbb{R}^2$: $(x,y) = x(1,0) + y(0,1)$.

(b) $S = \{1, x, x^2, \dots\}$ spans $F[x]$ by definition.

(c) Consider $F^{\mathbb{N}} = \operatorname{Maps}(\mathbb{N}, F)$, the space of sequences in F. Define the sequences $(a_n^{(1)}), (a_n^{(2)}), \dots$ as follows

$$a_n^{(i)} := \begin{cases} 1, & n = i \\ 0, & n \neq i. \end{cases}$$

One might expect that $S = \{(a_n^{(i)}) \mid i \in \mathbb{N}\}$ spans $F^{\mathbb{N}}$. The problem is that taking span involves only finite sums. In fact, $\operatorname{span}(S)$ is the subspace of sequences w/ finitely many non zero entries.

$\square$

Note: any superset of a spanning set is also a spanning set.

1.4 Bases and Dimension

1.4.1 Definition Let V be a vector space over F.

(a) The vectors $v_1, \dots, v_n \in V$ are **linearly independent** if $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ implies that $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

(b) A subset S of V is **linearly independent** if the vectors in any finite subset of S are linearly independent.

$\square$

Note:

- An infinite set could be independent.
- An independent set does not contain 0.
- $\emptyset$ is independent
- v is independent iff $v \neq 0$
- v, w are independent iff v, w are not parallel.
- Any subset of an independent set is independent.

1.4.2 Definition Let V be a vector space over F.

(a) The vectors $v_1, \dots, v_n \in V$ are **linearly dependent** if they are not independent.

(b) A subset S of V is **linearly dependent** if there exists a finite subset of S

consisting of vectors that are dependent.

□

Note: any superset of a dependent set is also dependent.

1.4.3 Proposition A subset S of a vector space V is dependent if and only if there exists $v \in S$ such that $v \in \operatorname{span}(S \setminus \{v\})$.

Proof. Exercise.

□

1.4.4 Definition A **basis** for a vector space V is a linearly independent spanning set.

□

The next proposition explains the importance of a basis.

1.4.5 Proposition Let B be a basis for a vector space V. Then every vector in V can be written uniquely as a linear combination of vectors in B.

Proof. Let $v \in V$. Since B spans V, there is a finite index set I, a collection of vectors $\{b_i \mid i \in I\} \subseteq B$, and a collection of scalars $\{\alpha_i \mid i \in I\} \subseteq F$ such

$$v = \sum_{i \in I} \alpha_i b_i .$$

Suppose there is another finite set J, another collection of vectors $\{b_j \mid j \in J\} \subseteq B$ and scalars $\{\beta_j \mid j \in J\}$, such that

$$v = \sum_{j \in J} \beta_j b_j .$$

Then

$$0 = v - v = \sum_{i \in I \setminus J} \alpha_i b_i - \sum_{i \in J \setminus I} \beta_i b_i$$

$$+ \sum_{i \in I \cap J} (\alpha_i - \beta_i) b_i$$

The set $\{b_i \mid i \in I \cup J\} \subseteq B$ is independent. Hence, $\alpha_i = 0$ for all $i \in I \setminus J$, $\beta_i = 0$ for all $i \in J \setminus I$, and $\alpha_i = \beta_i$ for all $i \in I \cap J$.

□

1.4.6 Example (bases)

(a) F^n has the following basis, called the standard basis:

$$e_1 := (1, 0, 0, \dots, 0)$$
$$e_2 := (0, 1, 0, \dots, 0)$$
$$\vdots$$
$$e_n = (0, 0, 0, \dots, 1).$$

(b) Similarly, $F^{m \times n}$ has a basis E_{ij} where E_{ij} is the matrix with precisely one nonzero entry equal to 1 in the ith row, j^{th} column.

(c) $B = \{1, x, x^2, \dots\}$ is a basis for $F[X]$,

(d) $B = \{1, x, x^2, \dots, x^n\}$ is a basis $F_n[x]$.

(e) $B = \emptyset$ is a basis for $V = \{0\}$.

□

Note: $F[x]$ does not have a finite basis.

Question: Does every vector space have a basis? For example, does $F^{\mathbb{N}}$ have a basis?

1.4.7 Lemma If B is independent and $B \cup \{s\}$ is dependent, then $s \in \operatorname{span}(B)$.

Proof. If $B \cup \{s\}$ is dependent, then there is a nontrivial linear combo $\alpha_0 s + \alpha_1 b_1 + \cdots + \alpha_n b_n = 0$. Clearly, $\alpha_0 \neq 0$ because is independent. Hence,

$$s = -\frac{1}{\alpha_0} \sum_{i=1}^{n} \alpha_i b_i \in \operatorname{span}(B).$$

□

1.4.8 Proposition Let B be a subset of a vector space V. Then the following are equivalent:

(a) B is a basis.

(b) B is a maximal independent set.

(c) B is a minimal spanning set.

Proof. ((a) $\Rightarrow$ (b)) Suppose B is a basis. Then B is independent. Suppose C is another indep. set such that $B \subsetneq C$. Choose $v \in C \setminus B$. Then

$$v \in \operatorname{span}(B) \subseteq \operatorname{span}(C \setminus \{v\}).$$

By Prop 1.4.3, C is dependent - a contradiction.
((b)$\Rightarrow$(a)) Suppose B is maximal independent. Suppose $v \in V \setminus \operatorname{span}(B)$. Then $B \cup \{v\}$ is independent by Lemma 1.4.7. This contradicts maximality of B.
((b)$\Leftrightarrow$(c)) Exercise. □

1.4.9 Proposition **(replacement)** Let V be a vector space. If $v_1, \dots, v_n \in V$ are independent and $S = \{s_1, \dots, s_k\}$ is a finite spanning set, then for all $i \in \{1, \dots, n\}$ there exists $j \in \{1, \dots, k\}$ such that the vectors

$$v_1, \dots, v_{i-1}, s_j, v_{i+1}, \dots, v_n$$

are also independent.

Proof. Suppose not. Then WLOG the statement is false for $i=1$. This means that for every $j \in \{1, \dots, k\}$, the vectors $s_j, v_2, \dots, v_n$ are dependent. Then $s_j \in \operatorname{span}\{v_2, \dots, v_n\}$ by Lemma 1.4.7. Thus,

$$v_1 \in \operatorname{Span}(S) \subseteq \operatorname{span}\{v_2, \dots, v_n\}.$$

This contradicts independence of $v_1, \dots, v_n$. □

1.4.10 Proposition Let V be a vector space. If $v_1, \dots, v_n \in V$ are independent and $S = \{s_1, \dots, s_k\}$

is a spanning set, then $n \le k$.

Proof. Using Prop 1.4.9, there exists $j_1 \in \{1, \dots, k\}$ such that $s_{j_1}, v_2, \dots, v_n$ are independent. By repeated application, there exists $j_2, \dots, j_n \in \{1, \dots, k\}$ such that $s_{j_1}, s_{j_2}, \dots, s_{j_n}$ are independent. By independence, these vectors are all distinct. Thus,

$$n = |\{s_{j_1}, s_{j_2}, \dots, s_{j_n}\}| \le |S| = k.$$

☐

1.4.11 Proposition Let V be a vector space. If V has a finite spanning set, then any independent set can be extended to a basis.

Proof. By Prop 1.4.10, any independent set is finite w/ cardinality $\le |S|$. Let B be any independent set. If B is maximal independent, then B is a basis by Prop 1.4.8. If not, then there exists $v \in V \setminus B$ such that $B \cup \{v\}$ is independent. If $B \cup \{v\}$ is maximal indep, we are done. If not, repeat. This process ends after finitely many steps since $|S|$ is finite.

☐

Note: Since $B = \emptyset$ is always independent, any vector space w/ a finite spanning set has a basis!

1.4.12 Proposition Let V be a vector space. If V has a finite spanning set, then any two bases for V have the same cardinality.

Proof. Suppose B and C are bases. Since B is spanning and C is independent, Prop 1.4.10 implies that $|C| \leq |B|$. Interchange the roles of B and C to get $|B| \leq |C|$.

$\square$

1.4.13 Definition Let V be an F-vector space. The dimension $\dim_F V$ of V is defined to be the cardinality of a basis for V.

$\square$

Note:
- By Prop 1.4.12, a vector space w/ a finite spanning set has finite dimension and $\dim_F V$ is well-defined.
- Even if V has no finite spanning, it is possible to prove that any two base have the same cardinality.

1.4.14 Example (dimension)

(a) $\dim_F \{0\} = 0$

(b) $\dim_F F^n = n$

(c) $\dim_F F[x] = |\mathbb{N}| =: \aleph_0$

(d) $\dim_F F^{m \times n} = mn$

(e) Recall: we have subfields $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

- $\dim_{\mathbb{C}} \mathbb{C} = 1$ - $\dim_{\mathbb{R}} \mathbb{C} = 2$ (a basis $\{1, i\}$)

- $\dim_{\mathbb{R}} \mathbb{R} = 1$ - $\dim_{\mathbb{Q}} \mathbb{R} = \infty$

(Suppose B is a $\mathbb{Q}$-basis for $\mathbb{R}$. If $B = \{b_1, \dots, b_n\}$ is finite, then $\mathrm{Span}(B) = \{\alpha_1 b_1 + \cdots + \alpha_n b_n \mid \alpha_i \in \mathbb{Q}\}$ is a countable set. So $\mathrm{span}(B) \subsetneq \mathbb{R}$.)

1.4.15 Proposition Suppose V is a finite-dimensional vector space w/ $\dim V = n$. Then

(a) Any independent set of size n is a basis.

(b) Any spanning set of size n is a basis.

Proof. Combine Prop 1.4.8 and Prop 1.4.10. ☐

1.4.16 Proposition Let V be a finite-dimensional vector space and $W \subseteq V$ a subspace. Then W is finite-dimensional w/ $\dim W \leq \dim V$. Moreover, $W = V$ if and only if $\dim W = \dim V$.

Proof. Since V has a finite spanning set, so does W. Hence W is finite-dimensional. Let B be a basis for W. Then B is independent in V, hence can be extended to a basis for V by Prop 1.4.11. Thus, $\dim W \leq \dim V$. If $W = V$, then they share a basis so $\dim W = \dim V$. If $\dim W = \dim V$, then any basis B for W is a basis for V by Prop 1.4.15(a). Thus, $W = \mathrm{Span}(B) = V$. ☐