4.4.5 Theorem Let $L: V \to V$ be a linear operator on a finite-dimensional F-vector space V and assume F contains the roots of $c_L(x)$. Then

$$V = V_{(\lambda_1)} \oplus \cdots \oplus V_{(\lambda_n)}$$

where $\lambda_1, \ldots, \lambda_n$ are the pairwise distinct eigenvalues of L. Further, $\dim V_{(\lambda_i)} =$ the algebraic multiplicity of λ_i.

Proof. Induction on the number of distinct eigenvalues. If $n=1$, then $c_L(x) = (x-\lambda_1)^{d_1}$. By Cayley-Hamilton,

$$0 = c_L(L) = (L - \lambda_1 I)^{d_1}$$

Thus,

$$V = \ker(0) = \ker(L - \lambda_1 I)^{d_1} \overset{\text{Lemma 4.4.4.}}{=} V_{(\lambda)}$$

Let $n \geq 1$ and assume the statement is true for any operator w/ $n-1$ distinct eigenvalues. Assume L has eigenvalues $\lambda_1, \ldots, \lambda_n$. Then

$$c_L(x) = \prod_{i=1}^{n} (x - \lambda_i)^{d_i}.$$

Set $f(x) = (x-\lambda_1)^{d_1}$ and $g(x) = \prod_{i=2}^{n} (x-\lambda_i)^{d_i}$. Then f and g are coprime and $f(L)g(L) = 0$ by Cayley-Hamilton. By Lemma 4.3.9,

$$\begin{aligned} V &= \ker f(L) \oplus \ker g(L) \\ &= V_{(\lambda_1)} \oplus \ker g(L). \end{aligned}$$

Set $L_1 = L|_{V(\lambda_1)}$ and $L_2 = L|_{\ker g(L)}$. Then $L = L_1 \oplus L_2$ and $c_L(x) = c_{L_1}(x)\, c_{L_2}(x)$. Notice:

① λ_1 is not an eigenvalue of L_2. If $v \in \ker g(L)$ s.t. $Lv = \lambda_1 v$. Then $0 = g(L)(v) = g(\lambda_1) v$. But $g(\lambda_1) \neq 0$, so $v = 0$.

② λ_1 is the unique eigenvalue of L_1 by lemma 4.4.3.

Together, ① + ② imply that $c_{L_1}(x) = (x-\lambda_1)^{d_1}$ and $c_{L_2}(x) = \prod_{i=2}^{n} (x-\lambda_i)^{d_i}$. Thus, $d_1 = \deg(c_{L_1}(x)) = \dim V_{(\lambda_1)}$ and by inductive hypothesis

$$\ker g(L) = V_{(\lambda_2)} \oplus \cdots \oplus V_{(\lambda_n)}$$

and $\dim V_{(\lambda_i)} = d_i$. This completes the proof.

□

Remark: By definition $V_{(\lambda)} = \bigcup_{i \geq 1} \ker(L - \lambda Id)^i$. Then we proved that

$$V_{(\lambda)} = \ker(L - \lambda Id)^d$$

where d = algebraic multiplicity of λ. We can even better:

4.4.6 Proposition Let λ be an eigenvalue of L. The smallest positive integer m such that

$$V_{(\lambda)} = \ker(L-\lambda \mathrm{Id}_V)^m$$

is the multiplicity of λ as a root of the minimal polynomial $m_L(x)$.

Proof. Consider $L|_{V(\lambda)} : V_{(\lambda)} \longrightarrow V_{(\lambda)}$. Since λ is the unique eigenvalue of $L|_{V_{(\lambda)}}$, we have $m_{L|_{V_{(\lambda)}}}(x) = (x-\lambda)^m$. Then

$$\begin{aligned}
V_{(\lambda)} = \ker(L-\lambda \mathrm{Id})^i &\iff V_{(\lambda)} \subseteq \ker(L-\lambda \mathrm{Id})^i \\
&\iff (L-\lambda \mathrm{Id})^i(V_{(\lambda)}) = \{0\} \\
&\iff (L-\lambda \mathrm{Id})^i|_{V_{(\lambda)}} = 0 \\
&\iff (L|_{V_{(\lambda)}} - \lambda \mathrm{Id}_{V_{(\lambda)}})^i = 0 \\
&\iff m_{L|_{V_{(\lambda)}}} \text{ divides } (x-\lambda)^i.
\end{aligned}$$

Thus, the smallest possible integer w/ the property is m. By Theorem 4.4.5, the multiplity of λ as a root of $m_L(x)$ coincides w/ the multipldy of λ as a root of $m_{L|_{V_{(\lambda)}}}$. $\square$

Recap: If $\lambda_1, \dots, \lambda_n$ are the distinct eigenvalues of L, then

① $V_{(\lambda_i)} \subseteq V$ is L-invariant

② $V = \bigoplus_{i=1}^{n} V_{(\lambda_i)}$

③ $\dim V_{(\lambda_i)} =$ algebraic multiplicity of λ_i

④ $\{0\} \subsetneq \ker(L-\lambda_i I) \subsetneq \ker(L-\lambda_i I)^2 \subsetneq \cdots \subsetneq \underset{\underset{V_{(\lambda_i)}}{\|}}{\ker(L-\lambda_i I)^m}$

where m is the mult. of λ_i as a root of $m_L(x)$.

□

Remark: • The mult. of λ in $m_L(x)$ is 1 if and only if $V_{(\lambda)} = E_\lambda$ by ④

• The geometric mult of λ equals the algebraic multiplicity if and only if $V_{(\lambda)} = E_\lambda$ by ③.

Thus, our characterizations of diagonalizable operators are both equivalent to the statement that $V_{(\lambda)} = E_\lambda$ for all λ. Thus, the material in the section is just a refinement of what we saw in the diagonalizable case.

4.4.7 Lemma Let λ be an eigenvalue of $L: V \to V$. If $i \geq 1$ is a positive integer such that

$$\ker(L-\lambda Id)^i = \ker(L-\lambda Id)^{i+1}.$$

then $V_{(\lambda)} = \ker(L-\lambda Id)^i$.

Proof. Homework.

□

4.4.8 Corollary Let λ be an eigenvalue of $L: V \to V$ and let m be the multiplicity of λ as a root of $m_L(x)$. Set $V_i = \ker(L - \lambda \mathrm{Id})^i$. Then we have an ascending chain of subspaces

$$\{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_m = V_{(\lambda)}$$

where the inclusions are strict. ☐

Remark: We have reduced the problem of understanding a linear operator $L: V \to V$ (on a finite-dimensional vectorspace over a field containing the roots of $c_L(x)$) to the problem of understanding the operators $L|_{V_{(\lambda)}}: V_{(\lambda)} \to V_{(\lambda)}$. The key to understand $L|_{V_{(\lambda)}}$ is to consider the strictly ascending chain of subspaces

$$\{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_m = V_{(\lambda)}.$$

from Corollary 4.4.8.

Idea: Use the ascending chain of subspaces to build a basis for $V_{(\lambda)}$ with respect to which the matrix for $L|_{V_{(\lambda)}}$ is as simple as possible. Below, we will sketch the procedure for finding such a basis.

Sketch: let $L : V \longrightarrow V$ be a linear operator and λ an eigenvalue. Set $V_i = \ker(L-\lambda \mathrm{Id})^i$. We have an ascending chain of subspaces

$$\{0\}_V := V_0 \underbrace{\subset}_{r_1} V_1 \underbrace{\subset}_{r_2} V_2 \subset \cdots \subset V_{m-1} \underbrace{\subset}_{r_m} V_m = V_{(\lambda)}$$

where m is the mult of λ as a root of $m_L(x)$.

Set $r_i = \dim V_{i+1} - \dim V_i = \dim(V_{i+1}/V_i)$, then

$$\begin{aligned}\dim V_{(\lambda)} = \dim V_m &= \dim(V_m/V_{m-1}) + \dim(V_{m-1}) \\ &= \dim(V_m/V_{m-1}) + \dim(V_{m-1}/V_{m-2}) + \dim(V_{m-2}) \\ &\;\vdots \\ &= \cdots \\ &= \sum_{i=1}^{m} \dim(V_i/V_{i-1}) = \sum_{i=1}^{m} r_i.\end{aligned}$$

Since $V_{i-1} \subset V_i$ for each i, we can find a basis for V_i by finding a basis for V_{i-1} and adding to it a collection of vectors

$$v_1, \ldots, v_{r_i} \in V_i$$

w/ the property that the cosets

$$v_1 + V_{i-1}, v_2 + V_{i-1}, \ldots, v_{r_i} + V_{i-1} \in V_i/V_{i-1}$$

form a basis for V_i/V_{i-1} (this is essentially a converse to Weekly 3.1). This is not difficult to prove. This implies the following:

Suppose for each $i = 1, \dots, m$ that

Key Fact #1

$$B_i = \{v_1, \dots, v_{r_i}\} \subseteq V_i$$

is chosen such that $v_1 + V_{i-1}, \dots, v_{r_i} + V_{i-1}$ is a basis for V_i / V_{i-1}. Then

$$B_1 \cup B_2 \cup \cdots \cup B_m$$

is a basis for V_m.

Our chain of subspaces

$$\{0\} = V_0 \underbrace{\subset}_{r_1} V_1 \underbrace{\subset}_{r_2} V_2 \subset \cdots \subset V_{m-1} \underbrace{\subset}_{r_m} V_m \boxed{= V_{(\lambda)}}$$

has a special feature: $r_1 \geq r_2 \geq \cdots \geq r_m$.

Set $K = L - \lambda \mathrm{Id}$. Suppose $v_1, \dots, v_{r_i} \in V_i$ are chosen such that $v_1 + V_{i-1}, \dots, v_{r_i} + V_{i-1}$ is a basis for V_i / V_{i-1}. Then

Key Fact #2

$$K(v_1), \dots, K(v_{r_i}) \in V_{i-1}$$

and

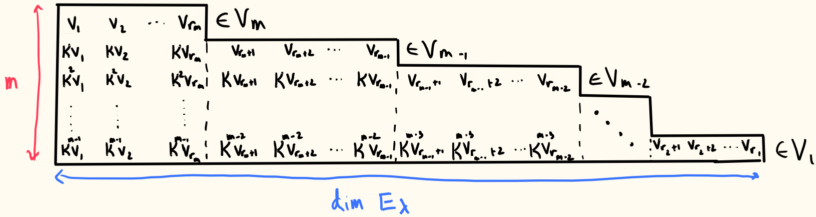
$$K(v_1) + V_{i-2}, \dots, K(v_{r_i}) + V_{i-2} \in V_{i-1}/V_{i-2}$$

are independent.

You will prove this fact in your Weekly 5. Thus,

$$r_{i-1} = \dim\left(V_{i-1}/V_{i-2}\right) \geq \dim\left(V_{i-1}/V_{i-2}\right) = r_i.$$

Key Facts 1 & 2 provide a procedure for building a basis for $V_m = V_{(\lambda)}$. We obtain a basis as follows:


$v_1 \quad v_2 \quad \cdots \quad v_{r_m}$ $\in V_m$
$Kv_1 \quad Kv_2 \quad Kv_{r_m}$ $\quad v_{r_m+1} \quad v_{r_m+2} \cdots v_{r_{m-1}}$ $\in V_{m-1}$
$K^2v_1 \quad K^2v_2 \quad K^2v_{r_m}$ $\quad Kv_{r_m+1} \quad Kv_{r_m+2} \cdots Kv_{r_{m-1}}$ $\quad v_{r_{m-1}+1} \quad v_{r_{m-1}+2} \cdots v_{r_{m-2}}$ $\in V_{m-2}$
$K^{m-1}v_1 \quad K^{m-1}v_2 \quad K^{m-1}v_{r_m}$ $\quad K^{m-2}v_{r_m+1} \quad K^{m-2}v_{r_m+2} \cdots K^{m-2}v_{r_{m-1}}$ $\quad K^{m-3}v_{r_{m-1}+1} \quad K^{m-3}v_{r_{m-1}+2} \cdots K^{m-3}v_{r_{m-2}}$ $\quad v_{r_2+1} \quad v_{r_2+2} \cdots v_{r_1}$ $\in V_1$
m
dim E_λ

One begins at row 1 with vectors $v_1, \dots, v_{r_m}$ such that

$$v_1 + V_{m-1}, \quad \dots, \quad v_{r_m} + V_{m-1}$$

form a basis for V_m / V_{m-1}. By Key Fact #2

$$Kv_1, \dots, Kv_{r_m} \in V_{m-1}$$

and

$$Kv_1 + V_{m-2}, \dots, Kv_{r_m} + V_{m-2} \in V_{m-1}/V_{m-2}$$

form an independent set in V_{m-1}/V_{m-2}. Extend to a basis

$$Kv_1 + V_{m-2}, \dots, Kv_{r_m} + V_{m-2}, v_{r_m+1} + V_{m-2}, \dots, v_{r_{m-1}} + V_{m-2}$$

for V_{m-1}/V_{m-2}. The vectors

$$Kv_1, Kv_2, \dots, Kv_{r_m}, v_{r_m+1}, \dots, v_{r_{m-1}}$$

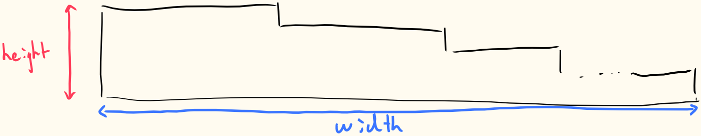
form the second row. Repeat!

By construction, the rows of the diagram satisfy the hypothesis of Key Fact #1:

1st row $\subseteq V_m$ $\ni$ provides a basis of V_m/V_{m-1}
2nd row $\subseteq V_{m-1}$ $\ni$ provides a basis of V_{m-1}/V_{m-2}
⋮
m^{th} row $\subseteq V_1$ $\ni$ provides a basis of $V_1/V_0 = V_1 = E_\lambda$.

Thus, the collection of vectors in the diagram forms a basis for $V_m = V_{(\lambda)}$.

The shape of the diagram is the array



which contains the basis for $V_{(\lambda)}$.

height $=$ # of rows $= m =$ mult. of λ as a root of $m_L(x)$

width $=$ # of columns $= \dim(E_\lambda) =$ geometric mult of λ.

area $= \dim(V_{(\lambda)}) =$ algebraic multiplicity of λ

The shape is independent of the choices we made in the procedure – it is completely determined by the dimensions of the subspaces

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{m-1} \subset V_m$$

using that $r_1 \geq r_2 \geq \cdots \geq r_m$.

# of columns of height $m = \dim(V_m/V_{m-1}) = r_m$

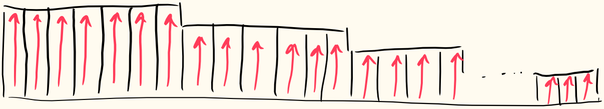
# of columns of height $m-1 = \dim(V_{m-1}/V_{m-2}) - \dim(V_m/V_{m-1})$

$$= r_{m-1} - r_m$$

⋮

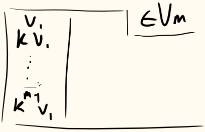
$$\# \text{ of columns of height } 1 = \dim(V_1/V_0) - \dim(V_2/V_1)$$
$$= r_1 - r_2$$

Now we need to order our basis such that the matrix is in a simple form. One way is to take the columns from left right with the vectors in each column ordered upwards:



$$B = (1^{st} \text{ column}, 2^{nd} \text{ colum}, \ldots, \text{Last column})$$
$$= (K^{m-1}v_1, K^{m-2}v_2, \ldots, Kv_1, v_1, K^{m-1}v_2, \ldots, v_{r_1})$$

Consider the first of the diagram



We have

- $(L - \lambda Id)(K^{m-1}v_1) = K^m v_1 = 0$ $\quad$ (since $v_1 \in V_m$)

$$\Rightarrow L(K^{m-1}v_1) = \lambda K^{m-1}v_1$$

- $(L - \lambda Id)(K^{m-2}v_1) = K^{m-1}v_1$

$$\Rightarrow L(K^{m-2}v_1) = \lambda K^{m-2}v_1 + K^{m-1}v_1$$

$\vdots$

- $(L - \lambda Id)(Kv_1) = K^2 v_1$

$$\Rightarrow L(Kv_1) = \lambda Kv_1 + K^2 v_1$$

- $(L-\lambda Id)(v_1) = Kv_1$

$$\Rightarrow \quad L(v_1) = \lambda v_1 + K v_1$$

Thus, we have computed the first several colums of matrix:

$$[L|_{V_{(\lambda)}}] = \left(\begin{array}{c|c} \begin{matrix} \lambda & 1 & 0 & \cdots & \cdots & \vdots \\ 0 & \lambda & 1 & 0 & \ddots & \vdots \\ \vdots & 0 & \lambda & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{matrix} & ? \\ \hline 0 & ? \end{array}\right)$$

In general, consider the jth column of heigh h in the basis diagram

$$\begin{bmatrix} x_j \\ Kx_j \\ K^2x_j \\ \vdots \\ K^{h-1}x_j \end{bmatrix} \qquad (x_j \in V_h)$$

which consists of h basis vectors. A similar computation shows that

$$[L|_{V_{(\lambda)}}] = \left(\begin{array}{c|c|c} & 0 & \\ \hline & \begin{matrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & \ddots & 1 \\ & & & & \lambda \end{matrix} & \\ \hline & 0 & \end{array}\right)$$

Notation:

$$J_h(\lambda) := \underbrace{\left.\begin{pmatrix} \lambda & 1 & & & & 0 \\ & \lambda & 1 & & & \\ & & \lambda & 1 & & \\ & & & \lambda & \ddots & \\ & & & & \ddots & 1 \\ 0 & & & & & \lambda \end{pmatrix}\right\} h}_{h}$$

The matrix $J_h(\lambda)$ is called the Jordan block of size h for λ. E.g.

$$J_1(\lambda) = (\lambda) \qquad J_2(\lambda) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad J_3(\lambda) = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

Ordering our basis in the above way yields

$$[L|_{V(\lambda)}] = \begin{pmatrix} J_m(\lambda) & & & & & & & & & 0 \\ & \ddots & & & & & & & & \\ & & J_m(\lambda) & & & & & & & \\ & & & J_{m-1}(\lambda) & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & J_{m-1}(\lambda) & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & J_2(\lambda) & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & J_2(\lambda) \\ & & & & & & & & & & J_1(\lambda) \\ & & & & & & & & & & & \ddots \\ 0 & & & & & & & & & & & & J_1(\lambda) \end{pmatrix}$$

a block diagonal matrix w/

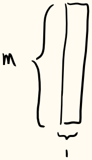
# of $J_m(\lambda)$ = \# of columns of height m in the shape

# of $J_{m-1}(\lambda)$ = \# of columns of height $m-1$ in the shape

$\vdots$

# of $J_1(\lambda)$ = \# of columns of height 1 in the shape.

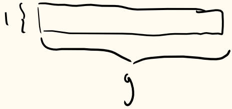
E.g. (some possible shapes)



geometric multiplicity = 1

algebraic multiplicity = $m \cdot 1 = m$

$J_m(\lambda)$ single Jordan block

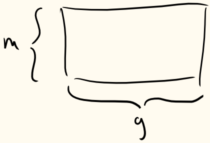


$m = 1$

geometric mult = $g = g \cdot 1$ = algebraic mult

All Jordan blocks have size 1

$$g\left\{\underbrace{\begin{pmatrix}\lambda & & \\ & \ddots & \\ & & \lambda\end{pmatrix}}_{g}\right. \quad \text{diagonal}$$

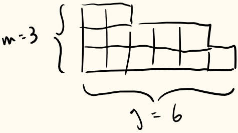


All Jordan blocks have the same size

Algebraic mult $= g \cdot m$

$= (\text{geometric mult}) \cdot m$

$$\begin{pmatrix}J_m(\lambda) & & \\ & \ddots & \\ & & J_m(\lambda)\end{pmatrix}$$



algebraic mult = 18

$$\begin{pmatrix}J_3(\lambda) & & & & & \\ & J_3(\lambda) & & & & \\ & & J_2(\lambda) & & & \\ & & & J_2(\lambda) & & \\ & & & & J_2(\lambda) & \\ & & & & & J_1(\lambda)\end{pmatrix} = \begin{pmatrix}\lambda & 1 & 0 & & & & & & & & & & & \\ 0 & \lambda & 1 & & & & & & & & & 0 & & \\ 0 & 0 & \lambda & & & & & & & & & & & \\ & & & \lambda & 1 & 0 & & & & & & & & \\ & & & 0 & \lambda & 1 & & & & & & & & \\ & & & 0 & 0 & \lambda & & & & & & & & \\ & & & & & & \lambda & 1 & & & & & & \\ & & & & & & 0 & \lambda & & & & & & \\ & & & & & & & & \lambda & 1 & & & & \\ & & & & & & & & 0 & \lambda & & & & \\ & & & & & & & & & & \lambda & 1 & & \\ & 0 & & & & & & & & & 0 & \lambda & & \\ & & & & & & & & & & & & \lambda\end{pmatrix}$$

In fact, the shape describes precisely the number and size of the Jordan blocks.

All of the above discussion was focused on finding a basis for $V_{(\lambda)}$ for which $[L|_{V_{(\lambda)}}]$ has a simple form.

Using the fact that $V = \bigoplus_{i=1}^{n} V_{(\lambda_i)}$ is an L-invariant direct sum decomposition of V, we $L = \bigoplus_{i=1}^{n} L|_{V_{(\lambda_i)}}$.

Hence,

$$[L] = \bigoplus_{i=1}^{n} [L|_{V_{(\lambda_i)}}].$$

Putting this all together, we obtain:

4.4.9 Theorem (Jordan Basis Theorem) Let V be a finite-dimensional vector space over a field F which contains the eigenvalues of $L : V \longrightarrow V$. Then there exists an ordered basis B of V such that

$$[L]_B = \begin{pmatrix} B_{\lambda_1} & & & 0 \\ & B_{\lambda_2} & & \\ & & \ddots & \\ 0 & & & B_{\lambda_n} \end{pmatrix} \quad \text{(block diagonal)}$$

$B_{\lambda_i} = [L|_{V_{(\lambda_i)}}]$

where $\lambda_1, \dots, \lambda_n$ are the distinct eigenvalues of L, and where each matrix B_{λ_i} is a block diagonal matrix consisting of Jordan blocks $J_\ell(\lambda_i)$ (of possibly varying sizes ℓ). In particular, $[L]_B$ is a direct sum of Jordan blocks for its eigenvalues. This matrix is called the Jordan Canonical Form and it is unique up to permutation of the blocks. Thus, L is uniquely determined by

- the distinct eigenvalues $\lambda_1, \dots, \lambda_n$; and
- the size and number of Jordan blocks for each eigenvalue. $\square$

4.4.10 Definition A Jordan matrix is a square matrix $A \in F^{n \times n}$ which is block diagonal consisting of Jordan blocks $J_d(\lambda)$ for various $d \in \mathbb{N}$, $\lambda \in F$.

4.4.11 Corollary A matrix $A \in F^{n \times n}$ is similar to a Jordan matrix if and only if F contains the eigenvalues of A. Such a matrix is the Jordan canonical form of A and is unique up to permutation of the blocks. Two matrices A, B satisfying the above condition are similar if and only if they have the same Jordan form. $\square$

4.4.12 Example

$$\begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix} \text{ and } \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix} \text{ are not similar}$$

$$\begin{matrix} \rotatebox{90}{=} \\ J_2(\lambda) \oplus J_3(\lambda) \end{matrix} \qquad \begin{matrix} \rotatebox{90}{=} \\ J_2(\lambda) \oplus J_1(\lambda) \oplus J_2(\lambda) \end{matrix}$$

$$\begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix} \text{ and } \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix} \text{ are similar.}$$

□

How to Compute the Jordan form

Let $A \in F^{n \times n}$.

① Compute $C_A(x)$. If $C_A(x)$ splits as a product of linear factors, then the Jordan form exists. Compute the eigenvalues.

② For each eigenvalue λ, compute the dimensions of the subspaces

$$\dim\left(\ker(A-\lambda I)\right) < \dim\left(\ker(A-\lambda I)^2\right) < \cdots < \dim\left(\ker(A-\lambda I)^m\right)$$

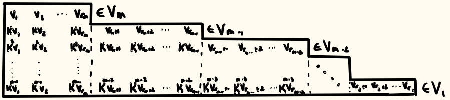
where m is the smallest positive integer s.t.

$$\dim\left(\ker(A-\lambda I)^m\right) = \dim V_{(\lambda)} = \text{alg. mult of } \lambda.$$

(Note: m = mult of λ in $m_A(x)$ so it could be helpful to compute $m_A(x)$).

The dimensions determine the shape of λ, i.e., the number and sizes of the Jordan blocks.

③ If you want to find P such that $P^{-1}AP = J$, then you must follow the procedure for constructing the basis

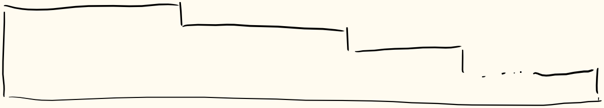


.

☐

Remark A basis of V putting A into Jordan form consists of generalized eigenvectors, but not every basis consisting of generalized eigenvectors will work. One must construct the basis in a special way, as described above.

Remark The characteristic and minimal polynomials of A put restrictions on the possible shapes



for each eigenvalue - but in general knowledge of the characteristic and minimal polynomials is not enough to determine the Jordan form. Basically, the polynomials determine the height and area, but not the shape.

4.4.13 Example Let $A = \begin{pmatrix} -12 & -13 & -17 \\ 27 & 29 & 36 \\ -13 & -14 & -17 \end{pmatrix} \in \mathbb{R}^{3\times 3}$.

Then $c_A(x) = (x-2)(x+1)^2$. The eigenvalues are $\lambda_1 = -1$ w/ alg. mult equal to 2 and $\lambda_2 = 2$ w/ alg. mult. equal to 1. One can check that $m_A(x) = c_A(x)$.

$\underline{\lambda_1 = -1}|$ We compute

$$\dim(\ker(A+I)) < \dim(\ker(A+I)^2) = \dim(\ker(A+I)^3)$$

$$\mathrm{Rref}(A+I) = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\dim E_{\lambda_1} = \dim(\ker(A+I)) = 1$$

The width of the shape is equal to 1.

$$\mathrm{Rref}((A+I)^2) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underset{\substack{\parallel \\ d_1}}{\dim V_{(\lambda_1)}} = \dim(\ker(A+I)^2) = 2$$

The area of the shape is equal to 2.

The height of the shape is 2 since λ_1 has mult. equal to 2 in $m_A(x)$. Thus, the shape for λ_1 is $\boxed{\begin{array}{c} \\ \hline \end{array}}$ which corresponds to a single Jordan block $J_2(-1)$.

To find the basis, choose a vector $b_2 \in \ker(A+I)^2$ but $b_2 \notin \ker(A+I)$ such as $b_2 := (1,0,-1)$. Then $b_1 := (A+I)b_2 = (6,-9,3)$. So $\boxed{\begin{array}{c} b_2 \\ b_1 \end{array}}$ is a basis for $V_{(\lambda_1)}$.

$\underline{\lambda_2 = 2}|$ We know now that $\dim V_{(\lambda_2)} = 1$ since $\dim V_{(\lambda_1)} = 2$ and $\dim \mathbb{R}^3 = 3$ (or since λ_2 has mult

1 in $m_A(x)$). Thus, the area of the shape is equal to 1 so the shape is $\square$. The shape corresponds to a single Jordan block $J_1(\lambda_2)$. A basis is any nonzero vector $b_3 \in V_{(\lambda_2)} = \ker(A-2I)$. We

$$\operatorname{Rref}(A-2I) = \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \end{pmatrix}$$

So we could take $b_3 = (1,-5,3)$ Thus, a basis for V which puts A into Jordan form is $B = (b_1, b_2, b_3)$. We have $J = P^{-1}AP$ where

$$J = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad P = T^{E}_{B} = \begin{pmatrix} 6 & 1 & 1 \\ -9 & 0 & -5 \\ 3 & -1 & 3 \end{pmatrix}$$

Note: J can be determined from the shapes alone, one does not need to compute P in order to determine J.

$\square$