Remark: An operator L is diagonalizable if and only if V decomposes as the direct sum of all the eigenspaces for L. Write $V = \bigoplus_{i=1}^{n} E_{\lambda_i}$. For each i, notice that $L(E_{\lambda_i}) \subseteq E_{\lambda_i}$. Thus, L restricts to a linear operator

$$L|_{E_{\lambda_i}} : E_{\lambda_i} \longrightarrow E_{\lambda_i}$$

In fact, $L|_{E_{\lambda_i}} = \lambda_i \, \mathrm{Id}_{E_{\lambda_i}}$ and $L = \bigoplus_{i=1}^{n} \lambda_i \, \mathrm{Id}_{E_{\lambda_i}}$.

Remark: We have seen that not every operator L is diagonalizable, i.e., V cannot be decomposed as a direct sum of eigenspaces for L. Ideally, we would like to find a simple matrix rep. for every operator. The remark suggests that we look for direct sum decompositions which are invariant under L.

4.2.16 Definition Let $L : V \longrightarrow V$ be a linear operator. A subspace W of V is L-invariant if $L(W) \subseteq W$. □

Remark: • If W is L invariant, then L restricts to a linear operator on W

$$L|_{W} : W \longrightarrow W$$

- Every subspace W of V has a complement W' so that $V = W \oplus W'$. However, there is no reason that any complement of W will be L-invariant.

Choose bases B and B' for W and W'. Then $B \cup B'$ is a basis for V. Say $|B| = k$ and $\dim V = n$. Then

$$[L]_{B\cup B'} = \begin{array}{cc} \begin{matrix} \overbrace{\qquad}^{k} & \overbrace{\qquad}^{n-k} \end{matrix} & \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \begin{matrix} \}k \\ \}n-k \end{matrix} \end{array}$$

where $A \in F^{k\times k}$, $B \in F^{k\times n-k}$, $C \in F^{n-k\times k}$ and $D \in F^{n-k\times n-k}$. Then W is L-invariant if and only if $C = 0$ and W' is L-invariant if and only if $B = 0$. So a direct sum decomp of V into L invariant subspaces yields a matrix in block diagonal form.

Note:

Remark: The remark above provides a strategy for understanding an operator L. Find a decomp

$$V = V_1 \oplus \cdots \oplus V_n$$

into L-invariant subspaces. Then study the operators

$$L|_{V_i} : V_i \longrightarrow V_i .$$

One can understand L by using the fact that $L = \bigoplus_{i=1}^{n} L|_{V_i}$. For diagonalizable, we can take $V_i = E_{\lambda_i}$.

4.3 The Minimal Polynomial

4.3.1 Theorem Let $L : V \to V$ be a linear operator on a finite-dimensional vectorspace V. There exists a monic polynomial $m_L(x) \in F[x]$ that is unique with the following property: $m_L(L) = 0$ and if $p(x) \in F[x]$ also satisfies $p(L) = 0$, then $m_L(x)$ divides $p(x)$. The polynomial $m_L(x)$ is called the **minimal polynomial** of L.

Proof. The most elegant proof uses ring theory. The ring $F[x]$ is a a principal ideal domain. - every ideal is generated by a single element. The set

$$I = \{ p(x) \in F[x] \mid p(L) = 0 \}$$

is an ideal in $F[x]$. We need to prove that $I \neq 0$, then there is a unique monic polynomial which generates I, and this is equivalent to the theorem statement. Consider the operators

$$1, L, L^2, \ldots, L^{n^2-1}, L^{n^2} \in \mathrm{End}(V) = F^{n \times n}$$

where $n = \dim V$. There are n^2+1 operators, so they must be dependent. There is a nontrivial linear combo

$$\sum_{i=0}^{n^2} \alpha_i L^i = 0$$

Thus, $p(x) = \sum_{i=0}^{n^2} \alpha_i x^i$ is a nonzero element of I $\square$

4.3.2 Theorem Let $L : V \to V$ be a linear operator on a finite-dimensional vector space V. Then $m_L(x)$ and $c_L(x)$ have the same roots (with possibly different multiplicities).

Proof. Suppose $m_L(\lambda) = 0$. Then $m_L(x) = (x-\lambda)p(x)$ for some $p(x)$. Since $\deg m_L(x) > p(x)$, $p(L) \neq 0$ by minimality of $m_L(x)$. So chose $v \in V$ s.t. $w := p(L)(v) \neq 0$. Then

composition of functions

$$0 = m_L(L) = (L - \lambda Id)\, p(L)$$

Evaluate at v:

$$\begin{aligned} 0 = m_L(L)(v) &= (L - \lambda Id)\, p(L)(v) \\ &= (L - \lambda Id)(w) \\ &= Lw - \lambda w. \end{aligned}$$

So $Lw = \lambda w$. But $w \neq 0$, so w is an eigenvector of L w/ eigenvalue λ. This proves that $c_L(\lambda) = 0$.

Conversely, suppose λ is an eigenvalue of L. Then $Lv = \lambda v$ for some $v \neq 0$.

Lemma: If $Lv = \lambda v$, then $p(L)(v) = p(\lambda)v$ for any polynomial $p(x) \in F[x]$.

Using the lemma for the polynomial $m_L(x)$, we get

$$0 = m_L(L)(v) = m_L(\lambda)v$$

But $v \neq 0$, so $m_L(\lambda) = 0$. $\square$

4.3.3 Theorem (Cayley-Hamilton) Let L be a linear operator on a finite-dimensional vector space. Then $c_L(L) = 0$.

Proof. Homework. $\square$

4.3.4 Corollary The minimal polynomial $m_L(x)$ divides the characteristic polynomial $c_L(x)$. $\square$

4.3.5 Corollary The minimal polynomial has degree at most $n = \dim(V)$. $\square$

4.3.6 Example Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 0 \end{pmatrix} \in (\mathbb{Z}_5)^{3\times 3}$

We compute $m_A(x)$. Find $c_A(x)$:

$$c_A(x) = \det\begin{pmatrix} x+4 & 3 & 0 \\ 0 & x+3 & 1 \\ 0 & 4 & x \end{pmatrix}$$

$$= (x+4)\begin{vmatrix} x+3 & 1 \\ 4 & x \end{vmatrix}$$

$$= (x+4)(x^2+3x+1)$$

$p(x) = x^2 + 3x + 1$

$p(1) = 1 + 3 + 1 = 0$

So $x+4$ divides $p(x)$.

$\Rightarrow p(x) = (x+4)^2$

$$= (x+4)^3$$

Since $m_A(x)$ divides $c_A(x)$, we know that there are three possibilities: $m_A(x) - \{x+4, (x+4)^2, (x+4)^3\}$. Clearly, $A+4I \neq 0$ and $(A+4I)^3 = 0$ by Cayley-Hamilton. So it suffices to compute $(A+4I)^2$. We have

$$(A+4I)^2 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{pmatrix}^2$$

$$= \begin{pmatrix} 0 & 2 & \cdot \\ - & \cdot & \cdot \\ - & - & \cdot \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

therefore $m_L(x) = C_L(x) = (x+4)^3$.

To determine if A is diagonalizable, we compute E_λ for $\lambda = 1$.

$$I - A = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 4 & 1 \\ 0 & 4 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 3 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

So $\operatorname{rank}(I-A) - 2$ and $\dim E_{\lambda_1} = 1 \neq 3$. So A is not diagonalizable. We will prove soon that A is not diagonalizable since $m_A(x)$ has a repeated root.

□

4.3.7 Proposition Suppose L is diagonalizable w/ pairwise distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Then

$$m_L(x) = \prod_{i=1}^{n} (x - \lambda_i).$$

I.e., the minimal polynomial has no repeated roots.

Proof. By Thm 4.2.15, write $c_A(x) = \prod_{i=1}^{n} (x-\lambda_i)^{d_i}$ where $d_i = \dim E_{\lambda_i}$. By Thm 4.3.2, $m_A(x)$ has the same roots. So $p(x) := \prod_{i=1}^{n} (x-\lambda_i)$ is the candidate for $m_A(x)$ of smallest degree. Let $v \in V$. Then $v = v_1 + \dots + v_n$ where $v_i \in E_{\lambda_i}$. Hence

$$p(L)(v) = \sum_{i=1}^{n} p(L)(v_i) = 0$$

Since

$$\begin{aligned} p(L)(v_i) = \prod_{i=1}^{n} (L-\lambda_i)(v_i) &= \prod_{j \neq i} (L-\lambda_j)(L-\lambda_i)(v_i) \\ &= \prod_{j \neq i} (L-\lambda_j)(Lv_i - \lambda_i v_i) \\ &= 0. \end{aligned}$$

thus, $p(x) = m_L(x)$.

$\square$

Remark: Let $L_1 : V_1 \longrightarrow V_1$ and $L_2 : V_2 \longrightarrow V_2$ be operators. Using U.P. of Direct sum (Weekly 2) there is a unique linear operator

$$L_1 \oplus L_2 : V_1 \oplus V_2 \longrightarrow V_1 \oplus V_2$$

such $(L_1 \oplus L_2)(v_1, v_2) = (L_1(v_1), L_2(v_2))$. With respect to appropriate bases

$$[L_1 \oplus L_2] = \begin{pmatrix} [L_1] & 0 \\ 0 & [L_2] \end{pmatrix} = [L_1] \oplus [L_2].$$

So $$c_{L_1 \oplus L_2}(x) = \det\left(xI - \begin{pmatrix} [L_1] & 0 \\ 0 & [L_2] \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} xI - [L_1] & 0 \\ 0 & xI - [L_2] \end{pmatrix}$$

$$= \det(xI - [L_1]) \det(xI - [L_2])$$

$$= c_{L_1}(x)\, c_{L_2}(x).$$

How about $m_{L_1 \oplus L_2}(x)$. Let $p(x) \in F[x]$. Then

then $$p([L_1 \oplus L_2]) = p\begin{pmatrix} [L_1] & 0 \\ 0 & [L_2] \end{pmatrix} = \begin{pmatrix} p([L_1]) & 0 \\ 0 & p([L_2]) \end{pmatrix}$$

So $p(L_1 \oplus L_2) = 0$ iff $p(L_1) = 0$ and $p(L_2) = 0$.

Thus, $$m_{L_1 \oplus L_2}(x) \mid p(x) \iff p(L_1 \oplus L_2) = 0$$
$$\iff p(L_1) = 0 \text{ and } p(L_2) = 0$$
$$\iff m_{L_1}(x) \mid p(x) \text{ and } m_{L_2}(x) \mid p(x).$$

So $m_{L_1 \oplus L_2}(x)$ is the least common multiple (lcm) of $m_{L_1}(x)$ and $m_{L_2}(x)$

This provides another proof of 4.3.7. Consider $\lambda I_n = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}$. Then $c_{\lambda I_n}(x) = (x-\lambda)^n$ and $m_{\lambda I_n} = (x-\lambda)$.

Consider a diagonal matrix

$$A = \begin{pmatrix} \lambda_1 I_{d_1} & & \\ & \ddots & \\ & & \lambda_n I_{d_n} \end{pmatrix}$$

Then $c_A(x) = \prod_{i=1}^{n} (x-\lambda_i)^{d_i}$. Thus,

$$m(x) = \operatorname{lcm}\left(m_{\lambda I_{d_1}}(x), \ldots, m_{\lambda I_{d_n}}(x)\right)$$
$$= \operatorname{lcm}(x-\lambda_1, \ldots, x-\lambda_n)$$
$$= \prod_{i=1}^{n} (x-\lambda_i).$$

□

Next goal: prove converse to 4.3.7.

4.3.8 Lemma Suppose $P, Q : V \to V$ are linear operators such that $P+Q = \mathrm{Id}_V$ and $PQ = 0$. Then

(a) $QP = 0$

(b) $\mathrm{im}\, P = \ker Q$

(c) $\ker P = \mathrm{im}\, Q$

(d) $V = \mathrm{im}(P) \oplus \ker(P)$ □

Proof. Exercise.

4.3.9 Lemma Let $L : V \to V$ be a linear operator on a finite-dimensional vector space. Let $f(x), g(x) \in F[x]$ be coprime polynomials such that $f(L)g(L) = 0$. Then

$$V = \ker(f(L)) \oplus \ker(g(L))$$

and $\ker(f(L))$, $\ker(g(L))$ are L-invariant.

Proof. Since f, g are coprime, there exists polynomials $p(x), q(x) \in F[x]$ such that

$$p(x)f(x) + q(x)g(x) = 1 \qquad \left(\begin{array}{c}\text{Bezout's} \\ \text{Lemma in} \\ F[x]\end{array}\right)$$

Define $P = q(L)g(L)$ and $Q = p(L)f(L)$. Then the above implies that $P + Q = \mathrm{Id}_V$. Moreover, $PQ = q(L)g(L)p(L)f(L) = q(L)p(L)f(L)g(L) = 0$. By the lemma (part (d)),

$$V = \mathrm{im}(P) \oplus \ker(P).$$

We claim $\operatorname{im}(P) = \ker f(L)$. Indeed,

$$\begin{aligned}\operatorname{im}(P) = \operatorname{im}(q(L)g(L)) &\subseteq \ker(f(L)) \\ &\subseteq \ker(f(L)p(L)) \\ &= \ker(Q) \\ &= \operatorname{im}(P).\end{aligned}$$

Similarly, one can show that $\ker(P) = \ker(g(L))$. So we have the decomp. of V as claimed. The subspaces $\ker(f(L))$ and $\ker(g(L))$ are L-invariant. For instance, if $v \in \ker(f(L))$, then

$$\begin{aligned}f(L)(L(v)) &= (f(L) \circ L)(v) \\ &= (L \circ f(L))(v) \\ &= L(0) = 0.\end{aligned}$$

Thus, $L(v) \in \ker(f(L))$. □

4.3.10 Theorem Let $L : V \to V$ be a linear operator on a finite-dimensional vector space. Then L is diagonalizable if and only if $m_L(x)$ has no repeated roots.

Proof. $(\Rightarrow)$ Prop 4.3.7.

$(\Leftarrow)$ Proof by induction on the number of pairwise distinct eigenvalues $\lambda_1, \dots, \lambda_n$. If $n = 1$, then $m_L(x) = x - \lambda_1$. Thus, $0 = m_L(L) = L - \lambda_1 \mathrm{Id}$. Thus,

$$V = \ker(0) = \ker(L - \lambda_1 \mathrm{Id}) = E_{\lambda_1}.$$

Let $n \geq 1$ and assume that any operator w/ $n-1$ distinct eigenvalues is diagonalizable. Let L be an operator with pairwise distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. By hypothesis,

$$\begin{aligned} m_L(x) &= \prod_{i=1}^{n} (x - \lambda_i) \\ &= (x-\lambda_1) \prod_{i=2}^{n} (x-\lambda_i). \end{aligned}$$

Set $f(x) = (x-\lambda_1)$ and $g(x) = \prod_{i=2}^{n} (x-\lambda_i)$ so that $m_L(L) = f(x)g(x)$. Thus, $0 = f(L)g(L)$. By Lemma 4.3.9,

$$\begin{aligned} V &= \ker(f(L)) \oplus \ker(g(L)) \\ &= E_{\lambda_1} \oplus \ker(g(L)) \end{aligned}$$

Since both subspaces are L-invariant, $L = L_1 \oplus L_2$ where $L_1 = L|_{E_{\lambda_1}}$ and $L_2 = L|_{\ker(g(L))}$. Thus,

$$c_L(x) = c_{L_1}(x) c_{L_2}(x).$$

The only eigenvalue L_1 is λ_1, so L_2 has eigenvalues $\lambda_2, \ldots, \lambda_n$. We need to show that λ_1 is not eigenvalue of L_2. Suppose $v \in \ker(g(L))$ such that $L(v) = \lambda_1 v$. Then

$$0 = g(L)(v) = g(\lambda_1) v$$

But $g(\lambda_1) \neq 0$, so $v = 0$. By inductive hypothesis, L_2 is diagonalizable so that

$$\ker(g(L)) = E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_n}.$$

This completes the proof.

$\square$

4.3.11 Example $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix} \in \mathbb{R}^{3\times 3}$

Compute the characteristic polynomial: $c_A(x) = (x-1)(x-2)^2$. So $m_A(x) \in \{(x-1)(x-2), c_A(x)\}$. By Cayley-Hamilton. $c_A(A) = 0$. We have

$$(A-I)(A-2I) = \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix} \begin{pmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This shows that $m_A(x) = (x-1)(x-2)$. Since $m_A(x)$ has no repeated roots, A is diagonalizable.

☐

4.4 Jordan Canonical Form

In this section, whenever we consider a linear operator $L : V \longrightarrow V$ on an F-vector space V we will always assume that F contains all the roots of $c_A(x)$. In that case, the characteristic polynomial will always split into a product of linear factors

$$c_L(x) = \prod_{i=1}^{n} (x-\lambda_i)^{d_i}.$$

Still the operator may not be diagonalizable since the geometric multiplicities could be smaller than the algebraic multiplicities.

A field F is algebraically closed if every nonconstant polynomial $p(x) \in F[x]$ has a root in F. By the division algorithm and induction, one can show that all of the roots are in F and $p(x)$ splits into a product of linear factors. So for instance, the results of this section apply to every linear operator $L: V \to V$ when V is a complex vector space. However, the hypothesis may still be satisfied when F is not algebraically closed. For example; consider $F = \mathbb{Z}_2$,

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in F^{2\times 2} \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in F^{2\times 2}$$

Then

$$C_A(x) = (x+1)^2 \quad \text{and} \quad C_B(x) = x^2 + x + 1.$$

So the results apply to A, but not to B since $C_B(0) = 1$ and $C_B(1) = 1$ ie. C_B has no roots in F).

Idea: With the above hypotheses, the only obstruction to diagonalization is that the eigenspaces could be too small to give a direct sum decomposition of V. We will replace E_λ by possibly larger subspaces $V_{(\lambda)}$ of "generalized eigenvectors". We will have

1. $E_\lambda \subseteq V_{(\lambda)}$, $\dim V_{(\lambda)} =$ algebraic mult of λ
2. $V = \bigoplus_{i=1}^{n} V_{(\lambda_i)}$

3. $V_{(\lambda)}$ is L-invariant.

We will get a direct sum decomposition $L = \bigoplus_{i=1}^{n} L|_{V_{(\lambda_i)}}$. the matrices for the $L|_{V_{(\lambda_i)}}$ (w/ respect to a certain basis) will be fairly simple.

4.4.1 Definition Let $L: V \rightarrow V$ be a linear operator and λ an eigenvalue of L. A **generalized eigenvector** for λ is a nonzero vector $v \in V$ such that

$$(L - \lambda Id_V)^i v = 0$$

for some $i \geq 1$. The **rank** of a generalized eigenvector v is the smallest positive integer i such that $(L - \lambda I)^i v = 0$. The set of all generalized eigenvectors for λ, together with zero vector, is denoted by $V_{(\lambda)}$. Note:

$$V_{(\lambda)} = \bigcup_{i \geq 1} \ker(L - \lambda Id)^i .$$

The set $V_{(\lambda)}$ is called the **generalized eigenspace** for λ. ☐

Observe: The rank 1 generalized eigenvectors are just the ordinary eigenvectors. So $E_\lambda \subseteq V_{(\lambda)}$.

4.4.2 Lemma Let $L: V \rightarrow V$ be a linear operator and λ an eigenvalue. Then $V_{(\lambda)}$ is an L-invariant subspace of V.

Proof. By def, $V_{(\lambda)} = \bigcup_{i \geq 1} \ker(L - \lambda Id)^i$. In general, the union of subspaces is not a subspace. But in this case, the union is over a nested chain of subspaces

$$E_\lambda \subseteq \ker(L-\lambda Id)^2 \subseteq \ker(L-\lambda Id)^3 \subseteq \cdots$$

One can easily show in this situation that the union is a subspace. To show that $V_{(\lambda)}$ is L-invariant, let $v \in V_{(\lambda)}$. Then $(L-\lambda Id)^i v = 0$ for some $i \geq 1$. Then

$$(L-\lambda Id)^i(L(v)) = ((L-\lambda Id)^i L)(v)$$

$$\xrightarrow{\text{Corollary 4.1.2}} = L(L-\lambda Id)^i(v)$$

$$= L(0) = 0.$$

So $L(v) \in V_{(\lambda)}$ □

4.4.3 Lemma Let $L: V \to V$ be a linear operator and let λ be an eigenvalue of L. Then λ is the unique eigenvalue of the operator

$$L|_{V_{(\lambda)}} : V_{(\lambda)} \to V_{(\lambda)}.$$

Proof. Clearly, λ is an eigenvalue since $E_\lambda \subseteq V_{(\lambda)}$. Suppose $0 \neq v \in V_{(\lambda)}$ s.t. $Lv = \mu v$ for some $\mu \in F$. Then $(L-\lambda Id)^i v = 0$ for some $i \geq 1$. Set $p(x) = (x-\lambda)^i$ so that $p(L)v = 0$. Then

$$0 = p(L)v = p(\mu)v = (\mu-\lambda)^i v.$$

Since $v \neq 0$, $(\mu-\lambda)^i = 0$. This forces $\mu = \lambda$. □

4.4.4 Lemma Let $L : V \to V$ be a linear operator and λ an eigenvalue. Then $V_{(\lambda)} = \ker(L - \lambda Id_V)^d$ where d is the algebraic multiplicity of λ.

(i.e., the chain of subspaces terminates after d-steps
$$\{0\} \subseteq E_\lambda \subseteq \ker(L - \lambda Id_V)^2 \subseteq \dots \subseteq \ker(L - \lambda Id)^d = V_{(\lambda)}\ )$$

Proof. The characteristic polynomial of $L|_{V_{(\lambda)}}$ is

$$C_{L|_{V_{(\lambda)}}}(x) = (x - \lambda)^{\dim(V_{(\lambda)})}$$

by Lemma 4.4.3. By Cayley-Hamilton,

$$\left(L|_{V_{(\lambda)}} - \lambda\right)^{\dim(V_{(\lambda)})} = 0 .$$

Let W be a complement of $V_{(\lambda)}$ so that $V = V_{(\lambda)} \oplus W$. Then with respect to appropriate bases,

$$[L] = \begin{pmatrix} [L|_{V_{(\lambda)}}] & B \\ 0 & D \end{pmatrix}$$

for some matrices B and D. Thus,

$$\begin{aligned} C_L(x) &= C_{L|_{V_{(\lambda)}}}(x)\, C_D(x) \\ &= (x - \lambda)^{\dim(V_{(\lambda)})}\, C_D(x). \end{aligned}$$

Thus, $\dim V_{(\lambda)} \leq d$ where d is the algebraic mult. of λ. Thus,

$$(L - \lambda Id)^d \Big|_{V_{(\lambda)}} = \left(L|_{V_{(\lambda)}} - \lambda Id\right)^d = 0 .$$

Thus,
$$V_{(\lambda)} \subseteq \ker(L - \lambda Id)^d \subseteq V_{(\lambda)} .$$

□