MATH 117: Daily Assignment 9 Solutions

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See the daily assignment webpage for due dates, templates, and assignment description. Try to explain your reasoning and justify your computations for every problem. You should not appeal to any theorems that we have not proved yet.

1. Read Section 3.4 on your own. You can skip the reading if you are already familiar with the symmetric group. Write each permutation below as a product of transpositions. Then compute the sign of each permutation.

(a) $(1, 3, 4, 2) \in S_7$

(b) $(1, 4, 5, 7, 6) \in S_7$

(c) $(1, 2)(3, 4, 5, 7, 6) \in S_7$.

Solution. Use Prop 3.4.13 to decompose the permutations into a product of transpositions. Then one can use the formula $\text{sgn}(\sigma) = (-1)^k$ where k is the number of transpositions in some (any) decomposition of σ. This formula can be deduced from the results in 3.4.

(a) We have $(1, 3, 4, 2) = (1, 2)(1, 4)(1, 3)$. So this is an odd permutation and $\text{sgn}(1, 3, 4, 2) = (-1)^3 = -1$.

(b) $(1, 4, 5, 7, 6) = (1, 6)(1, 7)(1, 5)(1, 4)$. So this is an even permutation and $\text{sgn}(1, 4, 5, 7, 6) = (-1)^4 = 1$.

(c) This permutation is the product of the first two, and sgn is a group homomorphism, so

$$\text{sgn}((1, 2)(3, 4, 5, 7, 6)) = \text{sgn}(1, 3, 4, 2)\text{sgn}(1, 4, 5, 7, 6) = -1.$$

This permutation is odd.

□

2. (a) Compute the determinant of $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 1 & 0 \end{pmatrix} \in (\mathbb{Z}_5)^{3\times 3}$ using the Liebniz formula.

(b) Compute the determinant of $B = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 3 & 4 \\ 1 & 0 & 1 \end{pmatrix} \in (\mathbb{Z}_5)^{3\times 3}$ using row operations.

(c) Compute the determinant of $C = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 1 & 1 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 3 & 1 & 0 & 1 \end{pmatrix}$ using Cofactor expansion[1] along the first row.

(d) Compute the determinant of

$$D = \begin{pmatrix} 1 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 1 \end{pmatrix}$$

without doing any additional work.

[1] I won't prove this because it takes too long. Feel free to use it whenever you need to compute determinants.

(e) Which matrices above are invertible, if any?

Solution. (a) We have

$$\begin{aligned}\det(A) &= \det\left(\begin{smallmatrix}1&1&3\\1&2&3\\3&1&0\end{smallmatrix}\right)\\ &= \operatorname{sgn}(1)\cdot 1\cdot 2\cdot 0 + \operatorname{sgn}(12)\cdot 1\cdot 1\cdot 0 + \operatorname{sgn}(23)\cdot 1\cdot 1\cdot 3 + \operatorname{sgn}(13)\cdot 3\cdot 2\cdot 3\\ &\quad + \operatorname{sgn}(123) 1\cdot 1\cdot 3 + \operatorname{sgn}(132)\cdot 3\cdot 1\cdot 3\\ &= 0+0-3-18+3+9\\ &= 1.\end{aligned}$$

(b) In each step below, I added a scalar multiple of one row to another, which doesnt change the determinant (because det is alternating!). We have

$$\det(B) = \det\left(\begin{smallmatrix}1&3&4\\2&3&4\\1&0&1\end{smallmatrix}\right) = \det\left(\begin{smallmatrix}1&3&4\\2&3&4\\0&2&2\end{smallmatrix}\right) = \det\left(\begin{smallmatrix}1&3&4\\0&2&1\\0&2&2\end{smallmatrix}\right) = \det\left(\begin{smallmatrix}1&3&4\\0&2&1\\0&0&1\end{smallmatrix}\right) = 2.$$

At the end, I used that the determinant of an upper triangular matrix is the product of the diagonal elements.

(c) Using cofactor expansion along the first row, we have

$$\det(C) = \left(\begin{smallmatrix}2&0&0&2\\1&1&3&4\\1&2&3&4\\3&1&0&1\end{smallmatrix}\right) = 2\det(B) - 2\det(A) = 2\cdot 2 - 2\cdot 1 = 2.$$

(d) We have $D = A \oplus B \oplus C$. Therefore,

$$\det(D) = \det(A)\det(B)\det(C) = 1\cdot 2\dot{2} = 4.$$

□

3. Repeat Weekly 3.4(ii) using Proposition 3.3.10. Did you compute the matrix correctly the first time?

Solution. Since L is given by left multiplication by a matrix $A = \left(\begin{smallmatrix}0&1\\4&2\end{smallmatrix}\right)$, we have $[L]_E = \left(\begin{smallmatrix}0&1\\4&2\end{smallmatrix}\right)$. Using Prop. 3.3.10, we know

$$\begin{aligned}[L\otimes L]_{E\otimes E} = [L]_E \otimes [L]_E &= \left(\begin{smallmatrix}0&1\\4&2\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}0&1\\4&2\end{smallmatrix}\right)\\ &= \begin{pmatrix} 0\cdot\left(\begin{smallmatrix}0&1\\4&2\end{smallmatrix}\right) & 1\cdot\left(\begin{smallmatrix}0&1\\4&2\end{smallmatrix}\right)\\ 4\cdot\left(\begin{smallmatrix}0&1\\4&2\end{smallmatrix}\right) & 2\cdot\left(\begin{smallmatrix}0&1\\4&2\end{smallmatrix}\right)\end{pmatrix}\\ &= \left(\begin{smallmatrix}0&0&0&1\\0&0&4&2\\0&4&0&2\\1&3&3&4\end{smallmatrix}\right).\end{aligned}$$

Yes, I was correct the first time. □