MATH 117: Daily Assignment 7 Solutions

WRITE YOUR NAME HERE

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See the daily assignment webpage for due dates, templates, and assignment description. Try to explain your reasoning and justify your computations for every problem. You should not appeal to any theorems that we have not proved yet.

1. Let $V = F^2$ where $F = \mathbb{Z}_5$. Denote the elements of $\mathbb{Z}_5$ by $0, 1, 2, 3, 4$. A basis for V is given by $B = ((1,2),(1,0))$. Define a linear map $L : V^* \to V^*$ as follows: for any linear functional $f \in V^*$, $L(f)$ is the linear functional satisfying $L(f)(a,b) = f(a+3b, 4b)$. Let B^* denote the dual basis of B.

(a) Compute $[L]_{B^*}^{B^*}$ using only the definitions and Theorem 3.1.4.

(b) Compute $[L]_{B^*}^{B^*}$ using Theorem 3.2.2.[1]. Make sure you get the same result as in (a).

Solution. Write $B^* = (\varphi_1, \varphi_2)$ for the dual basis of B. I am going to do all the computation in (a) using only the definition of the dual basis. I will not appeal to the formulas for the coordinate vectors from Theorem 3.2.2.

(a) By definition, φ_1 is the unique linear functional satisfying $\varphi_1(1,2) = 1$ and $\varphi_1(1,0) = 0$ and φ_2 is the unique linear functional satisfying $\varphi_2(1,2) = 0$ and $\varphi_2(1,0) = 1$. We can obtain formulas for φ_1, φ_2 by extending linearly. To do this, write $(a,b) \in V$ as a linear combination of the vectors in B and plug the result into φ_i. Solve the system of equations

$$(a,b) = \alpha(1,2) + \beta(1,0).$$

Comparing coefficients yields $\alpha = 3b$ and $\beta = a - \alpha = a + 2b$. Thus,

$$\varphi_1(a,b) = 3b\varphi_1(1,2) + (a+2b)\varphi_1(1,0) = 3b.$$

Similarly, $\varphi_2(a,b) = a + 2b$.

Now, let's compute the coordinate vector $[L(\varphi_1)]_{B^*}$. To do this, solve the system $L(\varphi_1) = \alpha\varphi_1 + \beta\varphi_2$. Evaluate both sides at the vectors in B. We obtain

$$\alpha = \alpha\varphi_1(1,2) + \beta\varphi_2(1,2) = L(\varphi_1)(1,2) = \varphi_1(2,3) = 3 \cdot 3 = 4.$$

and

$$\beta = \alpha\varphi_1(1,0) + \beta\varphi_2(1,0) = L(\varphi_1)(1,0) = \varphi_1(1,0) = 0.$$

Thus, $[L(\varphi_1)]_{B^*} = (4,0)$. A similar computation to solve $L(\varphi_2) = \alpha\varphi_1 + \beta\varphi_2$ yields $\alpha = \varphi_2(2,3) = 3$ and $\beta = \varphi_2(1,0) = 1$. Thus, $[L(\varphi_2)]_{B^*} = (3,1)$. We conclude that

$$[L]_{B^*} = \left(\begin{smallmatrix} 4 & 3 \\ 0 & 1 \end{smallmatrix}\right).$$

[1] You have to find a linear map $K : V \to V$ such that $K^* = L$ in order to invoke the theorem

(b) Note that $L = K^*$ where $K : V \to V$ is the linear map defined by $K(a,b) = (a+3b, 4b)$. We have

$$K(1,2) = (2,3) = 4 \cdot (1,2) + 3 \cdot (1,0)$$

and

$$K(1,0) = (1,0)$$

so that

$$[L]_{B^*} = [K^*_{B^*}] = ([K]_B)^T = \left(\begin{smallmatrix} 4 & 0 \\ 3 & 1 \end{smallmatrix}\right)^T = \left(\begin{smallmatrix} 4 & 3 \\ 0 & 1 \end{smallmatrix}\right).$$

$\square$