# MATH 117: Daily Assignment 6 Solutions

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August 12, 2023

See the daily assignment webpage for due dates, templates, and assignment description. Try to explain your reasoning and justify your computations for every problem. You should not appeal to any theorems that we have not proved yet.

**1.** Let $F = \mathbb{Z}_2$, $V = F_3[x]$ and $W = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in F^{2\times 2} : a + d = 0 \right\}$. Define a linear map $L : V \to W$ via

$$L\left(a + bx + cx^2 + dx^3\right) = \left(\begin{smallmatrix} b+c & c+d \\ c+d & b+c \end{smallmatrix}\right).$$

Complete the following steps (the point is that computations involving linear maps are best done using matrices).

(a) Compute a basis $B$ for $V$ and a basis $C$ for $W$. We have already seen bases for both, but you should make sure you can come up with them on your own.

(b) Compute $[L]_B^C$.

(c) Compute a basis for the column space of $[L]_B^C$ using methods from Section 2.9.

(d) What is the rank of $[L]_B^C$? What is the nullity of $[L]_B^C$? You should be able to compute the nullity without computing a basis for the nullspace.

(e) Recall that $C$ defines a coordinate isomorphism $\varphi_C : W \to F^3$, $w \mapsto [w]_C$. This isomorphism sends any basis for $\operatorname{im}(L)$ to a basis for the column space of $[L]_B^C$, and vice-versa. Compute a basis for $\operatorname{im}(L)$ by evaluating $\varphi^{-1}$ at the basis you found in part (c).

(f) Compute a basis for the null space of $[L]_B^C$. It's easy to find a spanning set using a standard trick. Explain using part (d) why it's actually a basis.

(g) The coordinate isomorphism $\varphi_B : V \to F^4$, $v \mapsto [v]_B$ sends any basis for $\ker(L)$ to a basis for the null space of $[L]_B^C$, and vice-versa. Compute a basis for $\ker(L)$ by evaluating $\varphi_B^{-1}$ at the basis you found in part (f).

*Solution.* (a) I will work with the bases $B = (1, x, x^2, x^3)$ and $C = (\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right))$ from Daily's 4 & 5.

(b) I will compute $[L]_B^C$ directly from the definition. We have $L(1) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ so $[L(1)]_C = (0, 0, 0)$. Next, $L(x) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ so $[L(x)]_C = (1, 0, 0)$. Thirdly, we have $L(x^2) = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ so that $[L(x^2)]_C = (1, 1, 1)$. And finally, $L(x^3) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ from which we obtain $[L(x^3)]_C = (0, 1, 1)$. We conclude that

$$[L]_B^C = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

(c) The column space o$[L]_B^C$ is the subspace of $F^3$ spanned by the columns, i.e. the subspace $\operatorname{span}((1, 0, 0), (1, 1, 1), (0, 1, 1))$ of $F^3$. Row reduce the matrix until it is reduced-row echelon form:

$$\left(\begin{smallmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{smallmatrix}\right) \overset{R_2 + R_3 \mapsto R_3}{\rightarrow} \left(\begin{smallmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right) \overset{R_1 + R_2 \mapsto R_1}{\rightarrow} \left(\begin{smallmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right).$$

Columns two and three of the reduced form are leading. Therefore, the set $\{(1,0,0),(1,1,1)\}$ containing columns two and three of $[L]_B^C$ forms a basis for the column space.

(d) Apply the Rank-Nullity Theorem to the linear map $F^4 \to F^3$ given by left multiplication by $[L]_B^C$. From (c), the rank is equal to two. By Rank-Nullity, the nullity of $[L]_B^C$ is also equal to two.

(e) The isomorphism $\varphi_C^{-1} : F^3 \to W$ is defined via $\varphi^{-1}(\alpha,\beta,\gamma) = \alpha\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) + \beta\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) + \gamma\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$. Thus, a basis for $\mathrm{im}(L)$ is given by $\{\varphi_C^{-1}(1,0,0), \varphi_C^{-1}(1,1,1)\} = \{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)\}$.

(f) Suppose $(x,y,z,w) \in \ker([L]_B^C)$. The reduced-row echelon form of $[L]_B^C$ has the same kernel (elementary row operations produce equivalent systems of equations). So we have

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Reading this equation yields dependency relationships among $x,y,z,w$. Specifically, $y + w = 0$ and $y + z = 0$. Thus,

$$(x,y,z,w) = (x,w,w,w) = x(1,0,0,0) + w(0,1,1,1).$$

This shows that $\{(1,0,0,0),(0,1,1,1)\}$ spans the nullspace of $[L]_B^C$. But it contains two vectors, and the nullity is equal to two, so it must be a basis.

(g) The isomorphism $\varphi_B^{-1} : F^4 \to V$ is defined via $\varphi_B^{-1}(\alpha,\beta,\gamma,\delta) = \alpha + \beta x + \gamma x^2 + \delta x^3$. Thus,

$$\{\varphi_B^{-1}(1,0,0,0).\varphi_B^{-1}(0,1,1,1)\} = \{1, x + x^2 + x^3\}$$

is a basis for the kernel of $L$.

□

A lot of the above could have been deduced without using matrices because working over $\mathbb{Z}_2$ made a lot of things trivial. But in general, the above solutions provides a sort of algorithm for computing bases for the image and kernel of a linear map.

**2.** Let $V = \mathbb{R}[x]$ and let $W = \{p(x)(1+x^2) : p(x) \in V\}$. Convince yourself that $W$ is a subspace. Then use the First Isomorphism Theorem to construct an isomorphism $V/W \to \mathbb{C}$ of $\mathbb{R}$-vector spaces.

*Solution.* A basis for $V = \mathbb{R}[x]$ is given by $B = \{x^n : n \in \mathbb{N}\}$. The convention here is that $x^0 := 1$. Define a function $f : B \to \mathbb{C}$ via $f(x^n) = i^n$ where $i$ is the complex number satisfying $i^2 + 1 = 0$. Then there is a unique linear map $L : V \to \mathbb{C}$ satisfying $L(x^n) = f(x^n) = i^n$. Explicitly, the value of $L$ on a polynomial $p(x) = \sum_{n=0}^m a_n x^n$ is the complex number $L(p(x)) = p(i) = \sum_{n=0}^m a_n i^n$.[1]

It is easy to see that $L$ is surjective: an $\mathbb{R}$-basis for $\mathbb{C}$ is the set $\{1, i\}$ and we have $L(1) = 1$ and $L(x) = i$. Thus $\mathrm{im}(L) = \mathrm{span}(1,i) = \mathbb{C}$. Moreover, $W = \ker(L)$. On one hand, if $p(x) \in W$, then $p(x) = q(x)(1+x^2)$ for some $q(x) \in V$. Thus, $L(p(x)) = q(i)(1+i^2) = q(i) \cdot 0 = 0$. On the other hand, suppose $p(x) \in \ker(L)$. Then $p(i) = 0$. We need to use some basis facts about polynomials real coefficients which you may or may not have seen before. One can show using

[1]Note: you could have defined $L$ directly using this definition, and then verified that it was a linear map. The advantage of my approach (using Theorem 2.3.1) is that my map is automatically linear.

the division algorithm that $p(i) = 0$ if and only if $x - i$ divides $p(x)$. Another theorem implies that the complex conjugate of any root is a root. So also $x + i$ divides $p(x)$. This just means that $p(x) = q(x)(x - i)(x + i) = q(x)(1 + x^2)$ for some $q(x) \in V$. Thus, $p(x) \in W$.

Since $\ker L = W$ and $L$ is surjective, the First Isomorphism Theorem implies that $V/W \cong \mathbb{C}$.

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