MATH 117: Daily Assignment 4 Solutions

Jadyn V. Breland

August 9, 2023

Some hints for this assignment are written in the footnotes. See the daily assignment webpage for due dates, templates, and assignment description. Try to explain your reasoning for every problem. You should be able to do each problem without appealing to theorems we have not proved yet.

1. For each part, you are given a vector space V over a field F with ordered bases B and C. Compute the transition matrix[1] T_B^C in each case.

(a) $F = \mathbb{Z}_3$, $V = F_2[x]$, $B = (1+x, 1+x^2, x+x^2)$, $C = (2, x^2, 1+x)$. Note: I am using the notation 0,1,2 for the elements of $F = \mathbb{Z}_3$. So for instance $1 + 2 = 0$ in F.

(b) $F = \mathbb{Z}_5$, $V = F^2$, $B = ((1,4),(2,4))$, $C = ((1,1),(2,1))$. Note: I am using the notation 0,1,2,3,4 for the elements of $F = \mathbb{Z}_5$. So for instance, $3 \cdot 4 = 2$ in F.

(c) $F = \mathbb{Z}_2$, $V = \left\{ \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right) \in F^{2\times 2} : a_{11} + a_{22} = 0 \right\}$, $B = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)\right)$, $C = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\right)$

Solution. (a) Let $E = (1, x, x^2)$. Then

$$\begin{aligned} T_B^C = (T_C^E)^{-1} T_B^E &= \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}. \end{aligned}$$

(b) Let $E = ((1,0),(0,1))$. Then

$$\begin{aligned} T_B^C = (T_C^E)^{-1} T_B^E &= \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 1 & 2 \\ 4 & 4 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 4 & 2 \\ 1 & 4 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 2 \\ 4 & 4 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right). \end{aligned}$$

(c) There's no standard basis E in this case, so you just have to work with the definitions. The columns are the coordinates for vectors in B with respect to the basis C. Computing each column amounts to solving a systtem of equations. In the end, you will get

$$T_B^C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

[1] You may want to take advantage of Proposition 2.4.2(c) in case V has a nicer basis. The method to compute the inverse of a matrix using row reduction works over any field.

2. Let $F = \mathbb{Z}_3$. How many isomorphisms are there from $F^2 \to F^2$?

Solution. There is a bijection between the set of all linear maps from $F^2 \to F^2$ and and the set $F^{2\times 2}$ which sends invertible linear maps to invertible matrices and vice-versa. Thus, the problem is to count the number of invertible matrices in $F^{2\times 2}$. The counting argument is as follows. Consider an invertible matrix A over F. Then A is invertible if and only if its columns are independent in F^2. The first column has 8 choice (the zero vector is dependent). After the first column has been chosen, the second column must be independent, i.e., not a scalar multiple of the first column. Thus, there are 6 choices for the second column. The number of invertible 2×2 matrices over F is $8 \cdot 6 = 48$. $\square$