MATH 117: Daily Assignment 3 Solutions

Carl Friedrich Gauss

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Some hints for this assignment are written in the footnotes. See the daily assignment webpage for due dates, templates, and assignment description.

1. Compute rank(L) and nullity(L) for each of the following linear maps. You must justify your reasoning.

(a) $L : F_2[x] \to F_2[x]$, $L(a + bx + cx^2) = b + 2cx$ where F is any field.[1]

(b) $L : F^{2\times 2} \to F$, $L(A) = \mathrm{tr}(A)$ where F is any field. Here, $\mathrm{tr}(A)$ denotes the *trace* of the matrix A, i.e., the sum of the element on the main diagonal.

Solution. We need to find a basis for the image and the kernel of each map.

(a) Suppose $p(x) = a + bx + cx^2 \in \ker(L)$. Then $b + 2cx = L(p(x)) = 0$. If $\mathrm{ch}(F) \neq 2$, then 2 is invertible and the equation implies $b = c = 0$. Thus, $\ker(L) = \{a \in F[x] : a \in F\}$. A basis for this space is $\{1\}$ and nullity$(L) = 1$. If $\mathrm{ch}(L) = 2$, then $2 = 0$ so the equation only implies that $b = 0$. In that case, $\ker(L) = \{a + cx^2 : a, c \in F\}$. A basis for this space is $\{1, x^2\}$ and nullity$(L) = 2$. Similarly, if $\mathrm{ch}(F) \neq 2$, then $\mathrm{im}(L) = \{b + 2cx : b, c \in F\}$. A basis is $\{1, x\}$ and rank$(L) = 2$. If $\mathrm{ch}(F) = 2$, then $\mathrm{im}(L) = \{b \in F[x] : b \in F\}$, a basis is $\{1\}$, and rank$(L) = 1$.

(a) The image of L is nonzero for any field F. Since $\mathrm{im}(L)$ is a subspace of F, $\mathrm{im}(L) = F$. Thus, rank$(L) = 1$. Suppose $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \ker(L)$. Then $a + d = 0$. Thus,

$$A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) = a\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) + b\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) + c\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right).$$

The independence of the vectors in the set

$$\left\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\right\}$$

is easily verified. Thus, nullity$(L) = 3$.

Notice that in all three cases, the rank plus the nullity is equal to the dimension of the domain of the map. This is the rank-nullity theorem, which we will prove later in the course. □

2. For each part, you are given a finite-dimensional F-vector space V with ordered basis B and a vector $v \in V$. Compute $[v]_B$.

(a) $V = \mathbb{R}^3$, $F = \mathbb{R}$, $B = ((1, 1, 2), (2, 3, 2), (1, 0, 1))$, $v = (-2, 1, 4)$.

(b) $V = F_2[x]$, F is any field with $\mathrm{ch}\, F \neq 2$[2], $B = (1+x, 1+x^2, x+x^2)$, $v = v(x) = a+bx+cx^2$ where $a, b, c \in F$.

[1] Hint: the answer depends on the characteristic of the field! Handle the case $\mathrm{ch}\, F = 2$ separately.

[2] This condition is required for the vectors in B to be independent.

(c) $V = \mathbb{C}$, $F = \mathbb{R}$, $B = (i, 1-i)$, $v = x + iy$. Here i denotes the imaginary unit in $\mathbb{C}$.

Solution. (a) By definition, $[(-2,1,4)]_B = (a,b,c)$ where $a,b,c \in \mathbb{R}$ are the unique numbers satisfying $a(1,1,2) + b(2,3,2) + c(1,0,1) = (-2,1,4)$. This is a system of equations represented by the matrix

$$\begin{pmatrix} 1 & 2 & 1 & -2 \\ 1 & 3 & 0 & 1 \\ 2 & 2 & 1 & 4 \end{pmatrix}$$

Row-reducing via WolframAlpha yields

$$\begin{pmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -\frac{5}{3} \\ 0 & 0 & 1 & -\frac{14}{3} \end{pmatrix}$$

Thus, $[(-2,1,4)]_B = \left(6, -\frac{5}{3}, -\frac{14}{3}\right)$.

(b) By definition, $[a + bx + cx^2]_B = (\alpha, \beta, \gamma)$ where $\alpha, \beta, \gamma \in F$ are the unique coefficients satisfying $\alpha(1+x) + \beta(1+x^2) + \gamma(x+x^2) = a + bx + cx^2$. Comparing coefficients of each power of x yields $\alpha + \beta = a$, $\alpha + \gamma = b$, and $\beta + \gamma = c$. The matrix for this system of equations is

$$\begin{pmatrix} 1 & 1 & 0 & a \\ 1 & 0 & 1 & b \\ 0 & 1 & 1 & c \end{pmatrix}.$$

We can row-reduce as usual as long as we never multiply a row by zero. We have

$$\begin{pmatrix} 1 & 1 & 0 & a \\ 1 & 0 & 1 & b \\ 0 & 1 & 1 & c \end{pmatrix} \overset{R_2 - R_1 \mapsto R_2}{\rightarrow} \begin{pmatrix} 1 & 1 & 0 & a \\ 0 & -1 & 1 & b-a \\ 0 & 1 & 1 & c \end{pmatrix}$$

$$\overset{R_2 + R_3 \mapsto R_3}{\rightarrow} \begin{pmatrix} 1 & 1 & 0 & a \\ 0 & -1 & 1 & b-a \\ 0 & 0 & 2 & c+b-a \end{pmatrix}$$

$$\overset{R_1 + R_2 \mapsto R_1}{\rightarrow} \begin{pmatrix} 1 & 0 & 1 & b \\ 0 & -1 & 1 & b-a \\ 0 & 0 & 2 & c+b-a \end{pmatrix}$$

$$\overset{\frac{1}{2}R_3 \mapsto R_3}{\rightarrow} \begin{pmatrix} 1 & 0 & 1 & b \\ 0 & -1 & 1 & b-a \\ 0 & 0 & 1 & \frac{c+b-a}{2} \end{pmatrix}$$

$$\overset{-R_2 \mapsto R_2}{\rightarrow} \begin{pmatrix} 1 & 0 & 1 & b \\ 0 & 1 & -1 & a-b \\ 0 & 0 & 1 & \frac{c+b-a}{2} \end{pmatrix}$$

$$\overset{R_1 - R_3 \rightarrow R_1}{\rightarrow} \begin{pmatrix} 1 & 0 & 0 & \frac{a+b-c}{2} \\ 0 & 1 & -1 & a-b \\ 0 & 0 & 1 & \frac{c+b-a}{2} \end{pmatrix}$$

$$\overset{R_2 + R_3 \mapsto R_2}{\rightarrow} \begin{pmatrix} 1 & 0 & 0 & \frac{a+b-c}{2} \\ 0 & 1 & 0 & \frac{a-b+c}{2} \\ 0 & 0 & 1 & \frac{c+b-a}{2} \end{pmatrix}$$

Notice that I had to divide by two - it wasn't a problem because $\mathrm{ch}(F) \neq 2$. Thus, we conclude that $[a + bx + cx^2] = \left(\frac{a+b-c}{2}, \frac{a-b+c}{2}, \frac{c+b-a}{2}\right)$. Notice that since $a,b,c \in F$ were arbitrary, we can define a linear map $F_2[x] \to F^3,\ a + bx + cx^2 \mapsto \left(\frac{a+b-c}{2}, \frac{a-b+c}{2}, \frac{c+b-a}{2}\right)$ which is an isomorphism of F-vector spaces.

(c) Since $x + yi = (x + y)i + x(1 - i)$, we can conclude that $[x + yi]_B = (x - y, x)$. Notice that since $x, y \in \mathbb{R}$ were arbitrary, we can define a linear map $\mathbb{C} \to \mathbb{R}^2, x + yi \mapsto (x + y, x)$ which is an isomorphism of $\mathbb{R}$-vector spaces.

$\square$

3. (a) Construct a linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ which maps the plane $W_1 = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ bijectively onto the plane $W_2\{(x, y, z) \in \mathbb{R}^3 : 3x + 2y + z = 0\}$.[3]

(b) Construct a linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ which maps the plane $W_3\{(x, y, z) \in \mathbb{R}^3 : x - y - z = 0\}$ bijectively onto the plane $W_4 = \{(x, y, z) \in \mathbb{R}^3 : x - 3z = 0\}$.

Solution. (a) One possible linear map is given by $L(a, b, c) = (a, b, -3a - 2b + c)$. Here's how I found it. Construct a basis for $\mathbb{R}^3$ which contains a basis for W_1, such as

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

According to Theorem 2.3.1., any linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ is uniquely determined by its image on a basis. If I want W_1 to be bijectively mapped onto W_2, then I need to ensure that my map sends a basis for W_1 to a basis for W_2. A basis for W_2 is given by $\{(1, 0, -3), (0, 1, -2)\}$. Now, define a linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ via $L(1, 0, 0) = (1, 0, -3)$, $L(0, 1, 0) = (0, 1, -2)$, and $L(0, 0, 1) = (0, 0, 1)$. Extending by linearity, the image on an arbitrary vector is $L(a, b, c) = (a, b, -3a - 2b + c)$.

(b) One possible linear map is given by $L(a, b, c) = (3b, a - b, a - c)$. A basis for $\mathbb{R}^3$ which contains a basis for W_3 is $\{(1, 1, 0), (1, 0, 1), (0, 0, 1)\}$. Notice that $L(1, 1, 0) = (3, 0, 1)$ and $L(1, 0, 1) = (0, 1, 0)$. Since $\{(3, 0, 1), (0, 1, 0)\}$ is a basis for W_2, we can conclude that L maps W_3 bijectively onto W_4.

Notice that in each case I forced L to be an isomorphism by sending the basis vector not in the first subspace to a vector not in the second subspace. $\square$

[3]Theorem 2.3.1 my be useful here.