MATH 117: Daily Assignment 3

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Some hints for this assignment are written in the footnotes. See the daily assignment webpage for due dates, templates, and assignment description.

1. Compute rank(L) and nullity(L) for each of the following linear maps. You must justify your reasoning.

(a) $L : F_2[x] \to F_2[x]$, $L(a + bx + cx^2) = b + 2cx$ where F is any field.[1]

(b) $L : F^{2\times 2} \to F$, $L(A) = \mathrm{tr}(A)$ where F is any field. Here, $\mathrm{tr}(A)$ denotes the *trace* of the matrix A, i.e., the sum of the element on the main diagonal.

Solution. Your solution can go here. □

2. For each part, you are given a finite-dimensional F-vector space V with ordered basis B and a vector $v \in V$. Compute $[v]_B$.

(a) $V = \mathbb{R}^3$, $F = \mathbb{R}$, $B = ((1, 1, 2), (2, 3, 2), (1, 0, 1))$, $v = (-2, 1, 4)$.

(b) $V = F_2[x]$, F is any field with $\mathrm{ch}\, F \neq 2$[2], $B = (1+x, 1+x^2, x+x^2)$, $v = v(x) = a+bx+cx^2$ where $a, b, c \in F$.

(c) $V = \mathbb{C}$, $F = \mathbb{R}$, $B = (i, 1 - i)$, $v = x + iy$. Here i denotes the imaginary unit in $\mathbb{C}$.

Solution. Your solution can go here. □

3. (a) Construct a linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ which maps the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ bijectively onto the plane $\{(x, y, z) \in \mathbb{R}^3 : 3x + 2y + z = 0\}$.[3]

(b) Construct a linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ which maps the plane $\{(x, y, z) \in \mathbb{R}^3 : x - y - z = 0\}$ bijectively onto the plane $\{(x, y, z) \in \mathbb{R}^3 : x - 3z = 0\}$.

Solution. Your solution can go here. □

4. (optional) Try to construct a choice function on $\mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$, the set of all nonempty subsets of the real numbers. Then give an informal argument (not a proof) for why it is not possible.

Solution. Your solution can go here. □

[1] Hint: the answer depends on the characteristic of the field! Handle the case $\mathrm{ch}\, F = 2$ separately.

[2] This condition is required for the vectors in B to be independent.

[3] Theorem 2.3.1 my be useful here.