MATH 117: Daily Assignment 2 Solutions

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Some hints for this assignment are written in the footnotes. See the daily assignment webpage for due dates, templates, and assignment description.

1. Let F be any field. A *Fibonacci sequence* in F is a function $f : \mathbb{N} \to F$ defined recursively by letting $f(0), f(1)$ be elements of F and then setting $f(n+2) = f(n+1) + f(n)$ for all $n \in \mathbb{N}$. Show that the set $\mathcal{F}$ of all Fibonacci sequences in F is a subspace of $F^{\mathbb{N}}$. Then compute the dimension of this space.

Solution. Clearly the zero sequence is Fibonacci. It's easy to check that $\mathcal{F}$ is closed under addition and scalar multiplication using induction.

Let me show you how to compute the dimension, which is the main point of this problem. Define Fibonacci sequences i, j via $i(0) = 1,\ i(1) = 0$ and $j(0) = 0,\ j(1) = 1$. Suppose that $\alpha i + \beta j = 0$. This is an equation of sequences, with the right-hand side being the zero sequence. Evaluate both sides at 0 to get $\alpha = 0$ and at 1 to get $\beta = 0$. This proves that the sequences are independent. Now, I claim that $\{i, j\}$ spans $\mathcal{F}$.I will prove that $f = f(0)i + f(1)j$. This is trivial for $n = 0, 1$. We proceed by "strong" induction. Let $n \in \mathbb{N}$ and assume that $f(k) = f(0)i(k) + f(1)j(k)$ for all $k \leq n$. Then

$$\begin{aligned} f(n+1) &= f(n) + f(n-1) && (f \text{ is Fibonacci}) \\ &= f(0)i(n) + f(1)j(n) + f(0)i(n-1) + f(1)j(n-1) && (\text{inductive hypothesis}) \\ &= f(0)(i(n) + i(n-1)) + f(1)(j(n) + j(n-1)) && (\text{vector space properties in } \mathcal{F}) \\ &= f(0)i(n+1) + f(1)j(n+1). && (i, j \text{ are Fibonacci}) \end{aligned}$$

This completes the proof. Thus, $\dim +F(\mathcal{F}) = 2$. Notice that $i + j$ is the usual Fibonacci sequence. ☐

2. For each part, determine whether the set of vectors S is a spanning set for the vector space V over the field F. If S is a spanning set, determine whether or not it is a basis[1]. Justify your answers.

(a) $F = \mathbb{Q}$, $V = \mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \in \mathbb{R} : a, b \in \mathbb{Q}\}$, $S = \{1 - \sqrt{2}, 4\}$.

(b) $F = \mathbb{Z}_2$, $V = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in F^{2\times 2} : a_{11} + a_{22} = 0 \right\}$, $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$

(c) $F = \mathbb{Z}_2$, $V = F_2[x] = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in F\}$, $S = \{1 + x, 1 + x^2\}$

Solution. (a) A basis for V is $B = \{1, \sqrt{2}\}$. The set spans V by definition, and if $a + b\sqrt{2} = 0$, then $b = 0$ because if not $\sqrt{2} = -\frac{a}{b}$ is rational. Hence, also $a = 0$. So the vectors are independent and $\dim_F(V) = 2$. Thus, if S is independent, then S is a basis. Suppose that $a(1 - \sqrt{2}) + b \cdot 4 = 0$. Then $(a + 4b) \cdot 1 - a\sqrt{2} = 0$. By independence of B, $a = 0$ and $a - 4b = 0$. It follows that $b = 0$. This proves that S is independent.

[1] Hint: use a Proposition from Section 1.4!

(b) The set S is spanning. A funny way to check is to realize that there are only sev 2×2 matrices over $\mathbb{Z}_2$ with trace equal to 0. Three belong to the set an the others are all obvious linear combinations of S. One proves the matrices are independent by supposing you have a linear combination equal to the zero matrix and comparing coefficients. You will immediately conclude that the linear combination is trivial.

(c) This is not a spanning set because the vector space has dimension equal to three, but there are only two vectors in the set. For instance, the polynomial 1 is not in the span of S.

$\square$

3. Let $M := \mathrm{Mag}_3(\mathbb{R})$ denote the set of 3×3 magic squares with entries from $\mathbb{R}$.

(a) Show that M is a subspace of $\mathbb{R}^{3\times 3}$.

(b) Find a basis for M.[2]

(c) What is the dimension of M?

Solution. (a) The zero matrix is magic. It is trivial to check that the sum $M + N$of magic squares $M, N$with magic sums m and n is a magic square with magic sum $m + n$ and the scalar multiple αM by $\alpha \in \mathbb{R}$ is magic with magic sum αm. Details left to the motivated student.

(b) A magic square satisfies a homogeneous system of 8 linear equations in 10 variable defined by the row, column, and diagonal sums. The variables are the entries of the matrix and the magic sum. The corresponding matrix which represents this systems of linear equations is

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

Using WolframAlpha to row reduce yields the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -2 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & -\frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

After typing the first matrix in LaTeX I used ChatGPT to convert my code to WolframAlpha's syntax. After WolframAlpha row reduced the matrix for me, I used ChatGPT to convert the wolfram alpha code back to LaTeXThis saved me a lot of time typing. This matrix represents another system of equations with *the same solution space* as the original

[2] You will need to solve a large system of equations - you may use a computer algebra system (CAS) to do this part of the computations.

one. Moreover, we can read the dimension of the kernel from the reduced form: it is equal to the number of non-leading columns, which is three. Thus, $\dim_{\mathbb{R}}(M) = 3$. Another advantage is that the equations yield nice dependency relations for the first 7 variables in terms of the other 3. In the end, for a magic square with entries $a, b, c, d, e, f, g, h, i$ and magic sum m, we obtain

$$\begin{aligned}
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &= \begin{pmatrix} -\frac{2}{3}m - i & -\frac{2}{3}m - h & \frac{1}{3}m + h + i \\ \frac{2}{3}m + h + 2i & -\frac{1}{3}m & -\frac{4}{3}m - 2i - h \\ -m - h - i & h & i \end{pmatrix} \\
&= -\frac{1}{3}m \begin{pmatrix} 2 & 2 & -1 \\ -2 & 1 & 4 \\ 3 & 0 & 0 \end{pmatrix} - i \begin{pmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 1 & 0 & -1 \end{pmatrix} - h \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.
\end{aligned}$$

The three magic squares on the right-hand side must therefore span the space of magic squares (assuming I made no computational mistakes). They also form a basis since we already know the dimension is equal to three.

$\square$