MATH 117: Daily Assignment 11 Solutions

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See the daily assignment webpage for due dates, templates, and assignment description. Try to explain your reasoning and justify your computations for every problem. You should not appeal to any theorems that we have not proved yet.

1. Let $F = \mathbb{Z}_5$. Compute the minimal polynomial for the the following matrices in from Daily 10. Use the minimal polynomial to determine if the matrix is diagonalizable. Compare with your solution to Daily 10.

(a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \in F^{3\times 3}$

(b) $B = \begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix} \in F^{2\times 2}$.

(c) $C = \begin{pmatrix} 1 & 1 & 0 & 4 \\ 2 & 4 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in F^{4\times 4}$

(d) $D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in F^{3\times 3}$

Solution. (a) We have $c_A(x) = (x+4)(x+3)^2$. Since the $m_A(x)$ divides $c_A(x)$ and has the same roots, we have

$$m_A(x) \in \{(x+4)(x+3), c_A(x)\}.$$

We see that

$$(A+4)(A+3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $m_A(x) = (x+4)(x+3)$. The matrix is diagonalizable because $m_A(x)$ has no repeated roots, which agree with my solutions to Daily 10.

(b) We have $c_B(x) = x^2 + 4 = (x+1)(x+4)$. Therefore $m_B(x) = c_B(x)$ and B is diagonalizable. This agrees with my solution to Daily 10.

(c) We have $c_C(x) = (x+4)^2(x^2+2)$. Note that x^2+2 has no roots in F, so it doesn't factor as a product of two linear polynomials. We can automomatically conclude that C is not diagonalizable using Theorem 4.2.15(d). Let's compute $m_A(x)$ anyways. We know that $m_C(x)$ divides $c_C(x)$ and has the same roots as $c_C(x)$. Therefore,

$$m_C(x) \in \{(x+4), (x+4)^2, (x+4)(x^2+2), c_C(x)\}.$$

But actually $(x+4)$ and $(x+4)^2$ are not possible because the operator is not diagonalizable. You can else check directly that A is not a root of those polynomials: clearly, $C + 4I \neq 0$

and the first entry of $(A+4I)^2$ is equal to 2 so it is not equal to the zero matrix either. We have

$$(A+4I)(A^2+2I) = \begin{pmatrix} 0&1&0&4\\ 2&3&1&1\\ 0&0&0&3\\ 0&0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&1&4\\ 0&0&0&1\\ 0&0&3&1\\ 0&0&0&3 \end{pmatrix} = \begin{pmatrix} 0&0&0&3\\ 0&0&0&0\\ 0&0&0&4\\ 0&0&0&0 \end{pmatrix}$$

which is nonzero. We conclude that $m_C(x) = c_C(x)$.

(d) We have $c_D(x) = (x+4)^2(x+1)$. Therefore.

$$m_D(x) \in \{(x+1)(x+4), c_D(x)\}.$$

We see that

$$(D+I)(D+4I) = \begin{pmatrix} 2&1&0\\ 0&0&1\\ 0&0&2 \end{pmatrix}\begin{pmatrix} 0&1&0\\ 0&3&1\\ 0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&1\\ 0&0&0\\ 0&0&0 \end{pmatrix}$$

We conclude that $m_D(x) = c_D(x)$.

□

2. We proved in the lectures that similar matrices have the same characteristic polynomial. Do similar matrices have the same minimal polynomial? Explain.

Solution. Yes. Suppose $B = P^{-1}AP$. Let $m_A(x) = \sum_{k=0}^{n} \alpha_i x^i$ be the minimal polynomial of A. Then

$$\begin{aligned} m_A(B) &= m_A(P^{-1}AP) \\ &= \sum_{k=0}^{n} \alpha_i (P^{-1}AP)^i \\ &= \sum_{k=0}^{n} \alpha_i P^{-1}A^iP \\ &= P^{-1}(\sum_{k=0}^{n} \alpha_i A^i)P \\ &= P^{-1}m_A(A)P \\ &= 0. \end{aligned}$$

Therefore $m_B(x)$ divides $m_A(x)$. By symmetry (since $(P^{-1})^{-1}BP^{-1} = A$), also $m_A(x)$ divides $m_B(x)$. This implies $m_A(x) = m_B(x)$. □

3. Determine whether the matrices $E = \begin{pmatrix} -8&-10&-1\\ 7&9&1\\ 3&2&0 \end{pmatrix} \in \mathbb{R}^{3\times 3}$ and $F = \begin{pmatrix} -3&2&-4\\ 4&-1&4\\ 4&-2&5 \end{pmatrix} \in \mathbb{R}^{3\times 3}$. are similar[1].

Solutions. The matrices both have the same trace (equal to 1), the same determinant (equal to -1), the same characteristic polynomial (equal to $(x-1)^2(x+1)$). Yet, $m_E(x) = (x-1)^2(x+1)$ while $m_F(x) = (x-1)(x+1)$. By Problem 2, they cannot be similar! □

4. Suppose that $A \in \mathbb{C}^{n\times n}$ satisfies $A^3 = A$. Is A diagonalizable? What about if $A \in F^{n\times n}$, where F is an arbitrary field?

[1]Hint: similar matrices have a lot of the same invariants - check those first!

Solution. By hypothesis, A is a root of the polynomial $p(x) = x^3 - x = x(x+1)(x-1)$. Therefore, $m_A(x)$ divides $p(x)$. It follows that $m_A(x)$ has no repeated roots, and A is diagonalizable over $\mathbb{C}$. In fact, the preceding argument works for any field F with characteristic not equal to 2.

However, suppose F is a field of characteristic equal to 2. Then $-1 = 1$. Hence, $p(x) = x(x+1)^2$. Now it is possible that the minimal polynomial of A has a repeated root. For instance, $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ has minimal polynomial $m_A(x) = (x+1)^2$, and yet

$$A^3 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = A.$$

$\square$