MATH 117: Daily Assignment 10 Solutions

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See the daily assignment webpage for due dates, templates, and assignment description. Try to explain your reasoning and justify your computations for every problem. You should not appeal to any theorems that we have not proved yet.

1. Let $F = \mathbb{Z}_5$. Determine which of the following matrices are diagonalizable. If the matrix is diagonalizable, find an invertible matrix P such that $P^{-1}AP$ is diagonal.

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix}$.

(c) $\begin{pmatrix} 1 & 1 & 0 & 4 \\ 2 & 4 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Solution. (a) Let $A = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{smallmatrix}\right)$. Then $xI_3 - A = xI_3 + 4A = \left(\begin{smallmatrix} x+4 & 0 & 0 \\ 0 & x+3 & 0 \\ 4 & 0 & x+3 \end{smallmatrix}\right)$. Since the matrix is lower triangular, one can see immediately (using cofactor expansion along the first row) that $c_A(x) = (x+4)(x+3)^2$. The eigenvalues are 1 and 2. We have

$$\operatorname{Rref}(I_3 - A) = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \text{ and } \operatorname{Rref}(2I_3 - A) = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right).$$

Thus, $\dim E_1 = 1$ and $\dim E_2 = 2$. Since the geometric multiplicities are equal to the algebraic multiplicities, the matrix is diagonalizable. Reading the above matrices yields a basis $B = ((4,0,1),(0,1,0),(0,0,1))$ consisting of eigenvectors. Setting $P = T_B^E$ yields

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = P^{-1}AP.$$

(b) Let $B = \left(\begin{smallmatrix} 2 & 4 \\ 3 & 3 \end{smallmatrix}\right)$. Then $\operatorname{tr}(B) = 0$ and $\det(B) = 4$ so that $c_B(x) = x^2 + 4$. Since $c_B(1) = 0$ and $c_B(4) = 0$, the eigenvalues are 1 and 4 with algebraic multiplicities equal to 1. Clearly, $(1,1)$ is an eigenvector with eigenvalue 1 and $(1,3)$ is an eigenvector with eigenvalue 4. So the eigenspaces have dimensions equal to1, the geometric and algebraic multiplies agree,

B is diagonalizable, and $B = ((1,1),(3,2))$ is a basis consisting of eigenvectors. Setting $P = T_B^E$ yields

$$\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = P^{-1}BP.$$

(c) Let $C = \left(\begin{smallmatrix} 1 & 1 & 0 & 4 \\ 2 & 4 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)$. Then

$$\begin{aligned} c_C(x) = \det(xI_4 - C) &= \left|\begin{smallmatrix} x+4 & 4 & 0 & 1 \\ 3 & x+1 & 4 & 4 \\ 0 & 0 & x+4 & 2 \\ 0 & 0 & 0 & x+4 \end{smallmatrix}\right| \\ &= (x+4)^2 \left|\begin{smallmatrix} x+4 & 4 \\ 3 & x+1 \end{smallmatrix}\right| \\ &= (x+4)^2(x^2+2). \end{aligned}$$

The polynomial $x^2 + 2$ has no roots in F, so the matrix is not diagonalizable.

(d) Set $D = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Then $c_D(x) = (x+4)^2(x+1)$. The eigenvalues are 1 and 4 with algebraic multiplicities equal to 2 and 1, respectively. We have

$$\mathrm{Rref}(I_3 - D) = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$$

so that $\dim E_1 = 1$. Since this is not equal to the algebraic multiplicity of 1, the matrix D is not diagonalizable.

$\square$

2. Let F be a field. Let $W = \{A \in F^{2\times 2} : c_A(x) = x^2 + \alpha \text{ for some } \alpha \in F\}$. Is W a subspace of $F^{2\times 2}$?

Solution. The characteristic polynomial of $A \in F^{2\times 2}$ is given by $x^2 - \mathrm{tr}(A) + \det(A)$, see Theorem 4.2.6. Thus, $A \in W$ if and only if $\mathrm{tr}(A) = 0$. It follows that $W = \ker(\mathrm{tr})$, which is indeed a subspace of $F^{2\times 2}$. $\square$