Weekly Assignment 5

WRITE YOUR NAME HERE
MATH 117: Advanced Linear Algebra

August 28, 2022

See the weekly assignment webpage for due dates, templates, and assignment description. Make sure to justify any claims you make. You may not appeal to any results that we have not discussed in class.

For Problems 1 & 2, $L : V \to V$ is a linear operator on a finite dimensional vector space V over a field F which contains the roots of the characteristic polynomial of L. For any positive integer $i \geq 1$ and eigenvalue $\lambda \in F$ for L, we set $V_i := \ker((L - \lambda \operatorname{Id}_V)^i)$ and $\dim V_i = r_i$. The first two problems were lemmas used in the sketch of the proof of the Jordan basis theorem.

1. (a) Suppose that $i \geq 1$ is a positive such that $V_i = V_{i+1}$. Prove that $V_{(\lambda)} = V_i$.

(b) In the lecture, we proved that the multiplicity m of λ in $m_L(x)$ is the smallest integer with the property that $V_{(\lambda)} = V_m$. Prove that the dimensions of the V_i strictly increase until we reach V_m:

$$\{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{m-1} \subsetneq V_m = V_{m+1} = \cdots$$

This proves a method for computing m - just compute $\dim(V_i)$ until the first time the dimension does not increase. That is, m is the smallest i such that $\dim(V_i) = \dim(V_{i+1})$.

Proof. Write your proof here. □

2. Set $K = L - \lambda \operatorname{Id}_V$. Assume $i \geq 2$. Suppose that $v_1, \ldots, v_{r_i} \in V_i$ are chosen such that the cosets $v_1 + V_{i-1}, \ldots, V_{r_i} + V_{i-1}$ form a basis for V_i / V_{i-1}.

(a) Prove that $K(v_1), \ldots, K(v_{r_i}) \in V_{i-1}$.

(b) Prove that the cosets $K(v_1) + V_{i-2}, \ldots, K(v_{r_i}) + V_{i-2}$ are independent in V_{i-1}/V_{i-2}.

Proof. Write your proof here. □

3. Let $L : V \to V$ be a linear operator on a finite dimensional inner product space V.

(a) Prove that there is a unique linear operator $L^* : V \to V$ satisfying $\langle L(x), y \rangle = \langle x L^*(y) \rangle$ for all $x, y \in V$.

(b) Prove that a subspace W of V is L-invariant if and only if $W^\perp$ is L^*-invariant.

Proof. Write your proof here. □

4. Let B be an ordered orthonormal basis for a finite dimensional inner product space V. Compute $[L^*]_B$.

Proof. Write your proof here. □