Weekly Assignment 4

WRITE YOUR NAME HERE
MATH 117: Advanced Linear Algebra

August 19, 2022

In this assignment, you will prove the Cayley-Hamilton theorem. Throughout, V is a finite-dimensional vector space and $L : V \to V$ is a linear operator. See the weekly assignment webpage for due dates, templates, and assignment description. Make sure to justify any claims you make. You may not appeal to any results that we have not discussed in class.

1. Suppose that W is an invariant subspace of V. Then we obtain by restriction a linear operator $L|_W : W \to W$. Prove that the characteristic polynomial of $L|_W$ divides the characteristic polynomial of L.

Proof. Write your proof here. □

2. For $v \neq 0$, define $\operatorname{span}(L, v) := \operatorname{span}\{L^i(v) : i \in \mathbb{N}_0\}$. Here $L^0 := \mathrm{Id}_V$.

(a) Prove that $\operatorname{span}(L, v)$ is an L-invariant subspace of V.

(b) Prove that there exists a largest integer k such that $B := \{v, L(v), \ldots, L^{k-1}(v)\}$ is independent. Moreover, show that B is a basis for $\operatorname{span}(L, v)$

Proof. Write your proof here. □

3. Let $v \neq 0$ and let B be the basis for $W := \operatorname{span}(L, v)$ from Problem 1. Define $m_{L,v}(x) = a_0 + a_1 x + \cdots + a_{k-1}x^{k-1} + x^k \in F[x]$ where $a_0, a_1, \ldots, a_{k-1}$ are the unique coefficients such that

$$a_0 v + a_1 L(v) + \cdots + a_{k-1}L^{k-1}(v) + L^k(v) = 0.$$

Since W is L-invariant, we obtain by restriction a linear operator $L|_W : W \to W$. The goal of this problem is to show that the characteristic polynomial of $L|_W$ is given by $(-1)^k m_{L,v}(x)$.

(a) Compute the matrix $xI_k - [L|_W]_B$.

(b) Show that $\det(xI_k - [L|_W]_B) = (-1)^k(a_0 + a_1 x + \cdots + a_{k-1}x^{k-1} + x^k)$ using induction. For the inductive step, start by using cofactor expansion along the first row.

Proof. Write your proof here. □

4. Use Problems 1-3 to prove the Cayley-Hamilton theorem: the linear operator L is a root of its characteristic polynomial, that is, $\operatorname{cp}_L(L)$ is the zero operator in $\operatorname{End}(V)$.

Proof. Write your proof here. □