Weekly Assignment 2

WRITE YOUR NAME HERE
MATH 117: Advanced Linear Algebra

July 31, 2022

Some hints for this assignment are written in the footnotes. See the weekly assignment webpage for due dates, templates, and assignment description. Make sure to justify any claims you make. You may not appeal to any results that we have not discussed in class.

1. Suppose that $V = U \oplus W$ is the direct sum of subspaces U and W. The projection onto U along W is the function $P : V \to V$ defined via $P(v) = u$ where $u \in U$ is the unique vector satisfying $v = u + w$ for some $w \in W$.

(a) Prove that P is linear.

(b) Prove that a linear operator $L : V \to V$ is a projection onto some subspace if and only if $L^2 = L$.

Proof. Write your proof here. □

2. Let V_1, V_2 be vector spaces over F. The direct sum $V_1 \oplus V_2$ comes with linear maps

$$\iota_1 : V_1 \to V_1 \oplus V_2,\ v_1 \mapsto (v_1, 0) \quad \text{and} \quad \iota_2 : V_2 \to V_1 \oplus V_2,\ v_2 \mapsto (0, v_2).$$

Let Z be any other vector space and let $L_1 : V_1 \to Z$ and $L_2 : V_2 \to Z$ be any other linear maps. Prove that there is a unique linear map $L : V_1 \oplus V_2 \to Z$ with the property that $L \circ \iota_1 = L_1$ and $L \circ \iota_2 = L_2$. See Remark 1.

Proof. Write your proof here. □

3. Let V_1, V_2, W_1, W_2 be vector spaces over F and let $L_1 : V_1 \to W_1$ and $L_2 : V_2 \to W_2$ be linear maps.

(a) Use the Universal Property of the Direct Sum (see Remark 1) to show that there is a unique linear map

$$L_1 \oplus L_2 : V_1 \oplus V_2 \to W_1 \oplus W_2$$

satisfying $(L_1 \oplus L_2)(v_1, v_2) = (L_1(v_1), L_2(v_2))$.

(b) Suppose additionally that V_1, V_2, W_1, W_2 are finite-dimensional with ordered bases $B_1 = (a_1, \ldots, a_k)$, $B_2 = (b_1, \ldots, b_l)$, $C_1 = (c_1, \ldots, c_m)$, and $C_2 = (d_1, \ldots, d_n)$, respectively.

(i) Prove that $B := ((a_1, 0), \ldots, (a_k, 0), (0, b_1), \ldots, (0, b_l))$ is a basis for $V_1 \oplus V_2$. Similarly, $C := ((c_1, 0), \ldots, (c_l, 0), (0, d_1), \ldots, (0, d_n))$ is a basis for $W_1 \oplus W_2$.

(ii) Prove that the matrix for $L_1 \oplus L_2$ with respect to B and C has the following block diagonal form:

$$[L_1 \oplus L_2]_B^C = \begin{pmatrix} [L_1]_{B_1}^{C_1} & 0 \\ 0 & [L_2]_{B_2}^{C_2} \end{pmatrix}.$$

Proof. Write your proof here. □

4. Let V be finite-dimensional vector space and W a subspace. Suppose that $\{b_1, \ldots, b_n\}$ is a basis for W and extend this to a basis $\{b_1, \ldots, b_k, b_{k+1}, \ldots, n\}$ for V using Proposition 1.4.11. Prove that the set of vectors $\{b_{k+1} + W, \ldots, b_n + W\}$ is a basis for the quotient space V/W.

Proof. Write your proof here. □

Remark 1. *In other words, L is the unique linear map making the following diagram commute*

$$\begin{tikzcd}
V_1 \arrow[dr, "\iota_1"'] \arrow[drr, bend left, "L_1"] & & \\
& V_1 \oplus V_2 \arrow[r, dashed, "\exists! L"] & Z \\
V_2 \arrow[ur, "\iota_2"] \arrow[urr, bend right, "L_1"'] & &
\end{tikzcd}$$

The property described in Problem 1 is usually called the Universal Property of the Direct Product. It provides a precise answer to the question "How do I define a linear map out of the direct sum of two vector spaces?". One simply defines linear maps L_1 and L_2 as in the statement. Your proof will provide the recipe for constructing the desired linear map L. Moreover, it establishes a bijection of sets

$$\mathrm{Hom}_F(V_1 \oplus V_2, Z) \longleftrightarrow \mathrm{Hom}_F(V_1, Z) \oplus \mathrm{Hom}_F(V_2, Z)$$

$$L \longmapsto (L \circ \iota_1, L \circ \iota_2)$$

However, notice that the domain and codomain are actually vector spaces. It can be shown that this bijection is a linear map, i.e., a vector space isomorphism. Note that all of this readily generalizes to the direct sum of finitely many vector spaces. Compare all of this with the discussion on the Universal Property of the Quotient Space from the lecture.