3.3.5 Example

$$T^n(V) = \mathrm{Mult}(V\times\cdots\times V, F) \cong \mathrm{Hom}_F(V\otimes\cdots\otimes V, F)$$
$$= (V\otimes\cdots\otimes V)^*$$

A multilinear n-form corresponds to a linear functional on $V\otimes\cdots\otimes V$.

$$\mathrm{Hom}(V\otimes V, F) \cong \mathrm{Bil}(V\times V, F)$$

$(n=2)$
$$(V\otimes V)^* \xleftarrow{\Phi} \mathrm{Hom}_F(V, V^*)$$

Define Φ as follows. Any linear map $k: V\to V^*$ determines a bilinear map

$$\begin{aligned} V\times V &\to F \\ (v,w) &\mapsto (k(v))(w). \end{aligned}$$

By U.P., there is a unique linear map $\Phi(k): V\otimes V \to F$ satisfying $(\Phi(k))(v\otimes w) = (k(v))(w)$. Exercise: show that Φ is an isomorphism.

3.3.6 Theorem

Let V and W be vector spaces w/ bases $B = \{b_i\}_{i\in I}$ and $C = \{c_j\}_{j\in J}$ where I and J are any sets. Then

$$B\otimes C := \{ b_i\otimes c_j \mid (i,j)\in I\times J\}$$

is a basis for $V\otimes W$.

Proof. (spans) Suffices to show that a simple tensor $v \otimes w$ is in the span of $B \otimes C$. Let $v \in V$ and $w \in W$. Then $v = \sum_{i \in I'} \alpha_i b_i$ and $w = \sum_{j \in J'} \beta_j c_j$ for some finite subsets $I' \subseteq I$ and $J' \subseteq J$. Then

$$\begin{aligned} v \otimes w &= \sum_{I'} \alpha_i b_i \otimes \sum_{J'} \beta_j c_j \\ &= \sum_{I'} \alpha_i \Big(b_i \otimes \sum_{J'} \beta_j c_j \Big) \\ &= \sum_{I'} \alpha_i \sum_{J'} \beta_j \, (b_i \otimes c_j) \\ &= \sum_{(i,j) \in I' \times J'} (\alpha_i \beta_j)(b_i \otimes c_j) \in \operatorname{span}(B \otimes C). \end{aligned}$$

(independence) Let $B^* = \{\varphi_i\}_{i \in I}$ and $C^* = \{\psi_j\}_{j \in J}$ be the dual bases (it's okay that I, J may not be finite). For each $(i,j) \in I \times J$, we have a bilinear map

$$\begin{aligned} V \times W &\longrightarrow F \\ (v, w) &\longmapsto \varphi_i(v) \psi_j(w). \end{aligned}$$

Using UP of TP, there is a unique linear map $\varphi_i \otimes \psi_j : V \otimes W \to F$ such that

$$\varphi_i \otimes \psi_j (v \otimes w) = \varphi_i(v) \psi_j(w).$$

Suppose

$$\sum_{(i,j) \in I' \times J'} \alpha_{ij} \, b_i \otimes c_j = 0$$

for some finite subsets $I' \subseteq I$ and $J' \subseteq J$.

Fix $(i_0, j_0) \in I' \times J'$. Then

$$
\begin{aligned}
0 &= \varphi_{i_0} \otimes \psi_{j_0} \Big(\sum_{(i,j)\in I'\times J'} \alpha_{ij}\, b_i \otimes c_j \Big) \\
&= \sum_{(i,j)\in I'\times J'} \alpha_{ij}\, \varphi_{i_0}(b_i)\, \psi_{j_0}(c_j) \\
&= \sum_{(i,j)\in I'\times J} \alpha_{ij}\, \delta_{i_0,i}\, \delta_{j_0,j} \\
&= \alpha_{i_0,j_0}.
\end{aligned}
$$

So $\alpha_{i_0,j_0} = 0 \quad \forall (i_0,j_0) \in I' \times J'$.

$\square$

3.3.7 Example If V, W are f.d w/ ordered base $B = (b_1, \dots, b_n)$ and $C = (c_1, \dots, c_m)$, then one can get an ordered basis for $V \otimes W$

$$B \otimes C = (b_i \otimes c_j)_{\substack{1 \le i \le n \\ 1 \le j \le m}}$$

ordered lexicographically.

$\square$

3.3.9 Corollary If V and W are finite-dimensional vector spaces, then

$$\dim V \otimes W = \dim V \, \dim W.$$

$\square$

3.3.9 Proposition If $L: V_1 \longrightarrow W_1$ and $K: V_2 \longrightarrow W_2$ are linear maps, then there is a unique linear map

$$L \otimes K : V_1 \otimes V_2 \longrightarrow W_1 \otimes W_2$$

Satisfying $L \otimes K(v_1 \otimes v_2) = L(v_1) \otimes L(v_2)$

$v_1 \in V_1$, $v_2 \in V_2$.

Proof. Exercise. $\square$

3.3.10 Proposition Assume the notation of 3.3.9

If B_1, B_2, C_1, C_2 are bases for V_1, V_2, W_1, W_2, respectively, then

$$[L \otimes K]^{C_1 \otimes C_2}_{B_1 \otimes B_2} = [L]^{C_1}_{B_1} \otimes [K]^{C_2}_{B_2}$$

where $\otimes$ on the right is the "Kronecker Product" of Matrices. If

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \in F^{n \times m}, \quad B = \begin{pmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & & \\ b_{p1} & \cdots & b_{pq} \end{pmatrix} \in F^{p \times q},$$

then

$$A \otimes B = \begin{pmatrix} a_{11} B & \cdots & a_{1m} B \\ \vdots & \ddots & \vdots \\ a_{n1} B & \cdots & a_{nm} B \end{pmatrix} \in F^{np \times mq}.$$

Proof. Exercise. $\square$

3.4 Permutations

3.4.1 Definition The symmetric group of order n is the set S_n of all bijective functions $\sigma : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$. Elements of S_n are also called permutations. It is a group under function composition.

□

3.4.2 Example (two line notation) One way to write down a permutation is w/ "two-line notation". For example,

$$S_4 \ni \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \begin{array}{l} \leftarrow \text{domain} \\ \leftarrow \text{range.} \end{array}$$

$$\sigma(1) = 1, \quad \sigma(2) = 3, \quad \sigma(3) = 2, \quad \sigma(4) = 4$$

□

3.4.3 Definition Let $\sigma \in S_n$. We call σ a k-cycle if there exist pairwise distinct elements $a_1, \dots, a_k \in \{1, \dots, n\}$ such that

$$\sigma(a_1) = a_2, \quad \sigma(a_2) = a_3, \quad \dots, \quad \sigma(a_{n-1}) = a_n, \quad \sigma(a_n) = a_1$$

In that case, we use cycle notation:

$$\sigma = (a_1, \dots, a_n).$$

Two cycles $\sigma = (a_1, \dots, a_n)$ and $\tau = (b_1, \dots, b_m)$ are disjoint if the sets $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_m\}$ are disjoint.

□

Remark

- **(a)** Composition is done from right to left, so $\sigma\tau$ means "apply τ first" and then "apply σ".
- **(b)** $(a_1, \dots, a_n) = (a_2, \dots, a_n, a_1) = \cdots$
- **(c)** $(a_1, \dots, a_n)^{-1} = (a_n, a_{n-1}, \dots, a_2, a_1)$
- **(d)** Disjoint cycles commute: $\sigma\tau = \tau\sigma$.
- **(e)** A 2-cycle is called a transposition.

3.4.4 Proposition Every permutation $\sigma \in S_n$ can be written as a product of disjoint cycles.

Proof. Exercise. $\square$

3.4.5 Example

two-line

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 & 6 & 4 \end{pmatrix}$$

cycle notation

$$\pi = (132)(456)$$

disjoint!

$\square$

3.4.6 Definition Let $\sigma \in S_n$. A pair $\{i, j\} \subseteq \{1, \dots, n\}$ is call an **inversion** of σ if

$$\frac{j-i}{\sigma(j)-\sigma(i)} < 0 .$$

$\square$

3.4.7 Example

(a) $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$ $\{2,3\}$ is an inversion

$= (2\,3)$ Since $\dfrac{3-2}{\pi(3)-\pi(2)} = \dfrac{3-2}{2-3} < 0$.

(b) $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 3 & 5 & 6 \end{pmatrix}$ $\{2,3\}$ and $\{2,4\}$ are

$= (2\,4\,3)$ inversions.

$\square$

3.4.8 Definition

Let $\sigma \in S_n$. Then σ is even if σ has an even number of inversions and odd otherwise. The parity $\mathrm{pty}(\sigma)$ is 0 if σ is even and 1 if σ is odd. The sign $\mathrm{sgn}(\sigma)$ of σ is defined to be

$$\mathrm{sgn}(\sigma) = (-1)^{\mathrm{pty}(\sigma)} = \begin{cases} 1, & \text{if } \sigma \text{ is even} \\ -1, & \text{if } \sigma \text{ is odd} \end{cases}$$

This defines a function $\mathrm{sgn} : S_n \longrightarrow \underbrace{\{1,-1\}}_{\text{a group under mult.}}$. $\square$

3.4.9 Example

A transposition is a permutation that swaps two elements, i.e., a 2-cycle (i,j) for some $i,j \in \{1,\dots,n\}$ w/ $i<j$. One can show that (i,j) has $2(j-i)-1$ inversions. Since this is an odd number, 2-cycles are odd and $\mathrm{sgn}(i,j) = -1$.

$\square$

3.4.10 Lemma Let $\sigma, \tau \in S_n$. Then

$$\mathrm{sgn}(\sigma) = \prod_{i<j} \frac{\tau(j)-\tau(i)}{\sigma(\tau(j))-\sigma(\tau(i))}.$$

Proof. Since τ is a bijection, the following function

set of all 2 elmt subsets of $\{1,\dots,n\}$

$$\Phi_\tau : \binom{\{1,\dots,n\}}{2} \longrightarrow \binom{\{1,\dots,n\}}{2}$$

$$\{i,j\} \longmapsto \{\tau(i), \tau(j)\}$$

is surjective and hence bijective. So

$$\prod_{i<j} \frac{\tau(j)-\tau(i)}{\sigma(\tau(j))-\sigma(\tau(i))} = \prod_{i<j} \frac{j-i}{\sigma(j)-\sigma(i)}$$

Since Φ_τ is bijective. Since Φ_σ is bijective the numerator and denominator are the same, except possible the signes. So the product is 1 or -1. But $\frac{j-i}{\sigma(j)-\sigma(i)}$ is negative iff $\{i,j\}$ is is an inversion. Hence, the product is 1 iff the number of inversions is even. $\square$

3.4.11 Example

Consider $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. Then

$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$.

$$\prod_{i<j} \frac{j-i}{\sigma(j)-\sigma(i)} = \frac{(2-1)(3-1)(3-2)}{(1-3)(2-3)(2-1)} = 1 = \operatorname{sgn}(\sigma)$$

$$\prod_{i<j} \frac{\tau(j)-\tau(i)}{\sigma\tau(j)-\sigma\tau(i)} = \frac{(1-2)(3-2)(3-1)}{(3-1)(2-1)(2-3)} = 1 .$$

□

3.4.12 Proposition If $\sigma, \tau \in S_n$, then

$$\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) .$$

That is, $\operatorname{sgn} : S_n \longrightarrow \{1,-1\}$ is a group homomorphism.

Proof.

$$\begin{aligned} \operatorname{sgn}(\sigma\tau) &= \prod_{i<j} \frac{j-i}{\sigma\tau(j)-\sigma\tau(i)} \\ &= \prod_{i<j} \frac{\tau(j)-\tau(i)}{\sigma\tau(j)-\sigma\tau(i)} \cdot \prod_{i<j} \frac{j-i}{\tau(j)-\tau(i)} \\ &= \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau). \end{aligned}$$

□

3.4.13 Proposition Every permutation $\sigma \in S_n$ can be written as a product of transpositions. Even a product of adjacent transpositions $(1,2), (2,3), \ldots, (n-1,n)$.

Proof. By Prop 3.4.4, it suffices to prove the claim for a cycle $\sigma = (a_1, \ldots, a_n)$. We have

$$\sigma = (a_1, a_n) \cdots \cdots (a_1, a_3)(a_1, a_2).$$

Adjacent transpositions: exercise.

$\square$

3.4.14 Corollary If σ is a product of k-transpositions, then $\operatorname{pty}(\sigma) = k \pmod 2$. $\square$

3.5 Alternating Multilinear Forms

3.5.1 Definition An n-form $\omega : V \times \cdots \times V \to F$ is called alternating if

$$\omega(v_1, \ldots, v_n) = 0 \quad \text{whenever} \quad v_i = v_j \text{ for some } i \neq j.$$

The set of all alternating n-forms is a subspace $\Lambda^n(V) \subseteq T^n(V)$. $\square$

3.5.2 Proposition Let $\omega \in \Lambda^n(V)$. Then

(a) $\omega(v_1, \ldots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \ldots, v_n) = -\omega(v_1, \ldots, v_n)$

(b) For any $\sigma \in S_n$,

$$\omega(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \operatorname{sgn}(\sigma)\, \omega(v_1, \ldots, v_n).$$

(c) $\omega(v_1,\dots,v_i+\alpha v_j,\dots,v_n)=\omega(v_1,\dots,v_i,\dots,v_n)$ for $i\neq j$.

Proof.

(a) $\omega(v_1,\dots,v_{i+1},v_i,\dots,v_n)+\omega(v_1,\dots,v_n)$

$$=\omega(v_1,\dots,v_{i+1}+v_i,v_i+v_{i+1},\dots,v_n)$$
$$=0 \quad (\text{since } \omega \text{ is alternating})$$

(b) Write $\sigma=t_1,\dots,\tau_n$ where $\tau_1,\dots,\tau_n$ are adjacent transpositions. Apply (a) repeatedly:

$$\omega(v_{\sigma(1)},\dots,v_{\sigma(n)})=\mathrm{sgn}(\tau_1)\,\omega(v_{\tau_2\cdots\tau_n(1)},\dots,v_{\tau_2\cdots\tau_n(n)})$$
$$\vdots$$
$$=\mathrm{sgn}(\tau_n)\cdots\mathrm{sgn}(\tau_1)\,\omega(v_1,\dots,v_n)$$
$$=\mathrm{sgn}(\sigma)\,\omega(v_1,\dots,v_n).$$

(c) $\omega(v_1,\dots,v_i+\alpha v_j,\dots,v_n)$

$$=\omega(v_1,\dots,v_n)+\alpha\,\omega\cancel{(v_1,\dots,}v_j,\dots v_n)$$
$$=\omega(v_1,\dots,v_n$$

since $i\neq j$ implies $\omega(v_1,\dots,v_j,\dots,v_n)=0$.

□

3.5.3 Proposition Suppose $v_1,\dots,v_n\in V$ are dependent. Then

$$w(v_1, \dots, v_n) = 0$$

for any $w \in \Lambda^n(V)$.

Proof. There is a nontrivial linear combination

$$v_j = \sum_{i \neq j} \alpha_i v_i .$$

Then for any $w \in \Lambda^n(V)$,

$$\begin{aligned} w(v_1, \dots, v_j, \dots, v_n) &= w(v_1, \dots, \sum_{i \neq j} \alpha_i v_i, \dots, v_n) \\ &= \sum_{i \neq j} \alpha_i \, w(v_1, \dots, v_i, \dots, v_n) \\ &= 0 . \end{aligned}$$

$\square$

3.5.4 Corollary If V is n-dimensional and $k > n$, then the only alternating k-form is the zero form.

Proof. Any collection of k-vectors will be a dependent set since $k > n = \dim V$. Apply 3.5.3.

$\square$

3.5.5 Proposition If V is n-dimensional, then

$$\dim \Lambda^k(V) \leq \binom{n}{k}.$$

Proof. Let $w \in \Lambda^k(V)$. Then there is a unique linear map $\overline{w} : V \otimes \cdots \otimes V \to F$ corresponding to w using U.P. of the tensor product. Let $(b_1, \dots, b_n)$ be a basis for V. Then

$$\{b_{i_1}\otimes b_{i_2}\otimes\cdots\otimes b_{i_k} \mid 1\le i_1,\dots,i_k\le n\}$$

is a basis for $\underbrace{V\otimes\cdots\otimes V}_{k\text{-times}}$. Then $\overline{\omega}$ is uniquely determined by its values on the basis. So ω is uniquely determined by the values

$$\omega(b_{i_1},\dots,b_{i_k}) \qquad 1\le i_1,\dots,i_k\le n.$$

Since this is zero whenever two indices are the same, we need only consider the indices $i_1,\dots,i_k$ that are pairwise distinct. By 3.5.2(b), we need only consider the values

$$\omega(b_{i_1},\dots,b_{i_k})$$

where $1\le i_1<i_2<\cdots<i_k\le n$. How many possibilities are there? There are $\binom{n}{k}$ of them.

$\square$

3.5.6 Corollary If $\dim V=n$, then $\dim\Lambda^n(V)\le 1$.

$\square$

3.5.7 Theorem If $\dim V=n$, then $\dim\Lambda^k(V)=\binom{n}{k}$.

Proof. Let $B=(b_1,\dots,b_n)$ be a basis for V and let $v_1,\dots,v_k\in V$. Assume

$$[v_1]_B=(\alpha_{11},\alpha_{12},\dots,\alpha_{1n}) = (\varphi_1(v_1),\dots,\varphi_n(v_1))$$
$$[v_2]_B=(\alpha_{21},\alpha_{22},\dots,\alpha_{2n}) = (\varphi_1(v_2),\dots,\varphi_n(v_2))$$
$$\vdots$$

$$[v_k]_B = (\alpha_{k1}, \alpha_{k2}, \dots, \alpha_{kn}) = (\rho_1(v_k), \dots, \rho_n(v_k))$$

where $B^* = (\rho_1, \dots, \rho_n)$ is the basis dual to B.

Let $\mathcal{I}$ denote the set of increasing functions from $\{1, \dots, k\}$ to $\{1, \dots, n\}$. Note that $|\mathcal{I}| = \binom{n}{k}$. For each $g \in \mathcal{I}$, define $\omega_g : V \times \cdots \times V \to F$ as follows:

$$\begin{aligned} \omega_g(v_1, \dots, v_k) &= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} \alpha_{(g \circ \sigma)(i), i} \\ &= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} \rho_i(v_{(g \circ \sigma)(i)}) . \end{aligned}$$

The claim is that $\{\omega_g \mid g \in \mathcal{I}\}$ is a basis for $\Lambda^k(V)$. By Prop 3.5.5, it suffices to check that

- ω_g is an alternating k-form for all $g \in \mathcal{I}$
- $\{\omega_g \mid g \in \mathcal{I}\}$ is independent.

Both are painful exercises.

$\square$