Remark: V, V^*, V^{**} are all isomorphic because they all have the same dimension. Yet there is a sense in which V and V^{**} are more that just isomorphic - they are "naturally isomorphic", in contrast V and V^* are not naturally isomorphic.

Here's the idea: we obtain an isomorphism $V \xrightarrow{\sim} V^*$ by choosing a basis for V. There is no canonical basis. So there is no canonical isomorphism, every isomorphism involves an arbitrary choice. OTOH the isomorphism $V \xrightarrow{\sim} V^{**}$ was defined w/out specifying a basis at all. So, it is much more "natural".

The "unnaturallity" of $V \xrightarrow{\sim} V^*$ has the following consequence: there is no choice of a family of isomorphisms $\Phi_V : V \longrightarrow V^*$, one for each f.d vector space V, that are compatible w/ linear maps $L : V \longrightarrow W$ in the sense that the following diagram commutes:

$$\begin{tikzcd}
V \arrow[r, "\Phi_V"] \arrow[d, "L"'] \arrow[dr, phantom, "G"] & V^* \\
W \arrow[r, "\Phi_W"'] & W^* \arrow[u, "L^*"']
\end{tikzcd}$$

$\boxed{L^*(f) = f \circ L \text{ for } f \in W^*}$

Any construction of Φ_V involves a choice of basis for each V, and since L is arbitrary, there is no reason why the diagram would commute. OTOH: the family of isomorphisms $V \xrightarrow{\sim} V^{**}$ is natural!

3.1.9 Proposition Let $L: V \longrightarrow W$ be a linear map of finite-dimensional vector spaces. There is a linear map

$$L^{**}: V^{**} \longrightarrow W^{**}$$
$$(f: V^* \to F) \longmapsto f \circ L^*.$$

For any finite-dimensional vector spaces V and W and any linear map $L: V \longrightarrow W$, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\sim"] \arrow[d, "L"'] \arrow[dr, phantom, "G"] & V^{**} \arrow[d, "L^{**}"] \\ W \arrow[r, "\sim"'] & W^{**} \end{tikzcd}$$

Proof. Let $v \in V$. The claim is that

$$L^{**}(ev_v) = ev_{L(v)}.$$

Both sides of eq. are elements of W^{**}. Let $g \in W^*$. Then

$$\left(L^{**}(ev_v)\right)(g) = (ev_v \circ L^*)(g)$$
$$= ev_v(g \circ L)$$

$$= (g \circ L)(v).$$

OTOH:

$$ev_{L(v)}(g) = g(L(v)) = (g \circ L)(v).$$

Thus, the diagram commutes. One also needs to check that L^{**} is actually a linear map $V^{**} \longrightarrow W^{**}$.

$\square$

Remark: In the 1940's – 1950's, a mathematician Saunders MacLane wanted to understand this notion of "naturality".

To precisely define "natural isomorphism"

(ex: the family of

$$\begin{tikzcd} V \arrow[r, "\text{iso}"] & V^{**} \\ v \arrow[r, mapsto] & ev_v \end{tikzcd}$$

)

he had to define "functor"

(ex:

$$\begin{tikzcd} V \arrow[dd, "L"'] & & V^{**} \arrow[dd, "L^{**}"] \\ & {} \arrow[r, mapsto] & {} \\ W & & W^{**} \end{tikzcd}$$

)

and to define functor he had to define "category"

(ex: f.d vector spaces (objects)
linear maps (morphisms))

The branch of mathematics called "category theory" was born.

$\square$

3.2 The Dual of a Linear Map

3.2.1 Definition Let $L: V \longrightarrow W$ be a linear map. The linear map

$$L^*: W^* \longrightarrow V^*, \quad f \longmapsto f \circ L$$

is called the dual of L. Other names: transpose or adjoint. Exercise to check that L^* is linear. □

3.2.2 Theorem Let V and W be finite-dim. vector spaces with bases $B = (b_1, \dots, b_n)$ and $C = (c_1, \dots, c_m)$. Let $B^* = (\beta_1, \dots, \beta_n)$ and $C^* = (\psi_1, \dots, \psi_m)$ be the dual bases for V^* and W^*. Let $L: V \longrightarrow W$ be a linear map. Then

$$[L^*]^{B^*}_{C^*} = \left([L]^C_B\right)^T.$$

Proof. By def

$$\left([L]^C_B\right)^T = \left([L(b_1)]_C \cdots [L(b_n)]_C\right)^T$$

$$= \begin{pmatrix} [L(b_1)]_C \\ \vdots \\ [L(b_n)]_C \end{pmatrix} =: \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

$$[L^*]^{B^*}_{C^*} = \left([L^*(\psi_1)]_{B^*} \cdots [L^*(\psi_m)]_{B^*}\right)$$

$$=: \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix}.$$

Then we claim that $a_{ij} = b_{ij}$ $\forall i,j$.

$$(1) \quad L(b_i) = \sum_{k=1}^{m} a_{ik} c_k$$

$$(2) \quad L^*(\psi_i) = \sum_{\ell=1}^{n} b_{\ell i} \rho_\ell$$

Evaluate (2) at b_j. Then

$$\begin{aligned} b_{ji} = \sum_{\ell=1}^{n} b_{\ell i} \delta_{\ell j} = \sum_{\ell=1}^{n} b_{\ell i} \rho_\ell(b_j) &= L^*(\psi_i)(b_j) \\ &= \psi_i(L(b_j)) \\ &\overset{(1)}{=} \sum_{k=1}^{m} a_{jk} \psi_i(c_k) \\ &= \sum_{k=1}^{m} a_{jk} \delta_{ik} = a_{ji}. \end{aligned}$$

This proves the claim.

$\square$

3.2.3 Proposition Let $L : V \longrightarrow W$ be a linear map w/ V and W finite-dimensional. Then

$$\operatorname{rank} L = \dim(\operatorname{im} L) = \dim(\operatorname{im} L^*) = \operatorname{rank} L^*$$

Proof. Exercise.

$\square$

Another Proof: For any matrix $A \in F^{m \times n}$, $\dim(\operatorname{row} A) = \dim(\operatorname{col}(A))$.

Left mult by A defines a linear map

$$F^n \xrightarrow{A\cdot} F^m.$$

By Thm 3.2.2, the map A^* is rep. by A^T.

$$\begin{tikzcd}
(F^m)^* \arrow[r, "A^*"] & (F^n)^* \\
F^m \arrow[u, "\wr"] \arrow[r, "A^T"'] \arrow[ur, phantom, "G"] & F^n \arrow[u, "\wr"']
\end{tikzcd}$$

$$\begin{aligned} \dim(\mathrm{row}(A)) = \dim(\mathrm{col}(A^T)) &= \dim(\mathrm{col}(A^*)) \\ &= \dim(\mathrm{col}(A)). \end{aligned}$$

□

3.3 Multilinear Maps

Motivation: In your last linear algebra class, you studied the determinant of a square matrix. It defines a function

$$\det : F^{n\times n} \longrightarrow F$$

It turns out to be extremely useful. You probably know several methods for computing it (e.g. cofactor expansion), but not a formal definition. Writing down an explicit formula is fairly involved and not necessarily illuminating.

A different approach is to think about the properties the determinant has and try to characterize it as the unique function

$$F^{n\times n} \longrightarrow F$$

w/ those properties. It is useful to think about the determinant as a function of the columns of a matrix

$$F^{n\times n} = \underset{\substack{\uparrow \\ \text{column } 1}}{F^n} \times \cdots \times \underset{\substack{\uparrow \\ \text{column } n}}{F^n}$$

$$\begin{aligned} \det : F^n \times \cdots \times F^n &\longrightarrow F \\ (c_1, \dots, c_n) &\longmapsto \det(c_1, \dots, c_n) \end{aligned}$$

From this viewpoint, det is:

(1) "multilinear" i.e. linear when restricted to a single column.

(2) "alternating" i.e. $c_i = c_j$, $i \neq j \Rightarrow \det(c_1, \dots, c_n) = 0$

It turns out that the space of all alternating multilinear maps

$$F^n \times \cdots \times F^n \longrightarrow F$$

is one-dimensional. There is unique alternating multilinear map sending $I_n \longmapsto 1$. This function is the determinant.

3.3.1 Definition Let $V_1, \dots, V_n$ and W be vector spaces over F. A multilinear map is a function

$$L : V_1 \times \cdots \times V_n \longrightarrow W$$

that is linear in each "coordinate":

$$L(v_1, \dots, \alpha v_i + \beta v_i', \dots, v_n) = \alpha L(v_1, \dots, v_i, \dots, v_n) + \beta L(v_1, \dots, v_i', \dots, v_n)$$

for all $i=1,\dots,n$.

Equivalently, for each $i=1,\dots,n$, the composition

$$V_i \xhookrightarrow{\subseteq} V_1\times\cdots\times V_n \xrightarrow{L} W$$

is a linear map. The set of all multilinear maps from $V_1\times\cdots\times V_n$ to W is denoted by

$$\mathrm{Mult}(V_1\times\cdots\times V_n, W). \qquad \square$$

Remark: In the definition, we are thinking of $V_1\times\cdots\times V_n$ as a set not as a vector space.

Remark: $\mathrm{Mult}(V_1\times\cdots\times V_n, W)$ is a subspace of $\mathrm{Maps}(V_1\times\cdots\times V_n, W)$.

Remark: Special cases:

- $(n=1)$ just a linear map $V\to W$ $\quad \mathrm{Hom}(V,W)=\mathrm{Mult}(V,W)$
- $(n=2)$ called "bilinear maps"

$$V_1\times V_2\to W \qquad \mathrm{Bil}(V_1\times V_2, W):=\mathrm{Mult}(V_1\times V_2, W)$$

- For a vector space V, a multilinear map

$$\underbrace{V\times\cdots\times V}_{n\text{-times}} \longrightarrow F$$

is called a multilinear n-form. The set of n-forms on V is denoted by

$$\Upsilon^n(V):=\mathrm{Mult}(V\times\cdots\times V, F)$$

1-forms are linear functionals $\Upsilon^1(V)=V^*$

2-forms are bilinear form

Remark: The idea of a multilinear map is a natural generalization of a linear. But it would be kind of annoying to have to build a theory of multilinear algebra from the groundup. Fortunately, there is a way to turn multilinear maps into linear maps on a certain vector space called the "tensor product". The process is invertible, thus "multilinear algebra" will be subsumed by linear algebra.

Idea: (for bilinear maps) Given two vector spaces V and W, the tensor product $V \otimes W$ will be a new vector space equipped w/ a special bilinear map

$$\begin{tikzcd}
V \times W \arrow[r, "- \otimes -"] & V \otimes W \\
(v, w) \arrow[r, mapsto] & v \otimes w
\end{tikzcd}$$

Moreover, $-\otimes-$ will be the "universal" bilinear map out of $V \times W$ in the following sense: given any other v.s. Z and any other bilinear map $B : V \times W \to Z$, there exists a unique linear map $L : V \otimes W \to Z$ such that $L(-\otimes-) = B$. In other words, the following diagram commutes

$$\begin{tikzcd}
V \times W \arrow[r, "B"] \arrow[d, "-\otimes-"'] \arrow[dr, phantom, "\circlearrowleft"] & Z \\
V \otimes W \arrow[ur, dotted, "\exists! L"'] &
\end{tikzcd}$$

This is the "universal Property of the Tensor Product". By defining $V \otimes W$ so that it has this property, we will the desired bijection

$$\begin{tikzcd}
\mathrm{Bil}(V \times W, Z) \arrow[r, "\sim"] & \mathrm{Hom}(V \otimes W, Z) \\
B \arrow[r, mapsto] & L \\
L(-\otimes-) & L \arrow[l, mapsto]
\end{tikzcd}$$

Recall: We have seen several other universal properties:

- U.P. of Free Vector Space $F^{(X)}$

$$\mathrm{Hom}(F^{(X)}, Z) \xrightarrow{\sim} \mathrm{Maps}(X, Z)$$

- U.P. of Quotient space

$$\mathrm{Hom}(V/W, Z) \xrightarrow{\sim} \{L \in \mathrm{Hom}(V, Z) \mid W \subseteq \ker L\}$$

- U.P. of Direct Sum

$$\mathrm{Hom}(V \oplus W, Z) \xrightarrow{\sim} \mathrm{Hom}(V, Z) \oplus \mathrm{Hom}(W, Z)$$

3.3.2 Definition Let V and W be vector spaces over F. The tensor product $V \otimes W$ of V and W is the F-vectorspace defined as follows:

$$V \otimes W := F^{(V\times W)} \big/ N$$

where:

- $F^{(V\times W)}$ is the free vector space w/ basis $V \times W$

  $$F^{(V\times W)} = \left\{ f : V \times W \to F \;\middle|\; f(v,w) = 0 \ \forall \text{ but finitely many } (v,w) \in V \times W \right\}$$

  $$= \bigoplus_{(v,w)\in V\times W} F$$

  An element $x \in F^{(V\times W)}$ is written as a formal linear combination of elements of $V \times W$

  $$x = \sum_{(v,w)\in V\times W} \alpha_{(v,w)} (v,w)$$

  where $\alpha_{(v,w)} = 0$ for all but finitely many (v,w).

- $$N := \operatorname{Span}\Big(\{ (v_1+v_2, w) - (v_1,w) - (v_2,w) \mid v_1, v_2 \in V, \ w \in W \}$$
  $$\cup \{ (v, w_1+w_2) - (v,w_1) - (v,w_2) \mid v \in V, \ w_1, w_2 \in W \}$$
  $$\cup \{ (\alpha v, w) - \alpha (v,w) \mid v \in V, w \in W, \alpha \in F \}$$
  $$\cup \{ (v, \alpha w) - \alpha (v,w) \mid v \in V, w \in W, \alpha \in F \} \Big).$$

Automatically, $V \otimes W$ is a vector space. It comes w/ a map

$$\begin{tikzcd}
V \times W \arrow[r, hook, "\subseteq"] & F^{(V\times W)} \arrow[r, two heads, "\pi"] & F^{(V\times W)}/N = V \otimes W \\
(v,w) \arrow[r, mapsto] & (v,w) \arrow[r, mapsto] & (v,w)+N
\end{tikzcd}$$

which we denote by $-\otimes- : V\times W \longrightarrow V\otimes W$. In other words, for $(v,w)\in V\times W$, $v\otimes w = (v,w)+N$ is just the coset w/ representative (v,w). The coset $v\otimes w$ is called a simple tensor. $\square$

3.3.3 Proposition Let V and W be vector spaces over F. For all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, $\alpha\in F$, we have

(a) $(v_1+v_2)\otimes w = v_1\otimes w + v_2\otimes w$

(b) $v\otimes(w_1+w_2) = v\otimes w_1 + v\otimes w_2$

(c) $(\alpha v)\otimes w = \alpha(v\otimes w) = v\otimes(\alpha w)$.

In other words, $-\otimes- : V\times W \longrightarrow V\otimes W$ is bilinear.

Proof. All are immediate consequences of the def.

(a) $(v_1+v_2)\otimes w = (v_1+v_2,w)+N$

$$= (v_1+v_2,w) + \underbrace{(v_1,w)+(v_2,w)-(v_1+v_2,w)}_{\in N} + N$$

$$= (v_1,w)+(v_2,w)+N$$

$$= (v_1,w)+N + (v_2,w)+N$$

$$= v_1\otimes w + v_2\otimes w.$$

Others are similar. $\square$

Remark: $F^{(V\times W)}$ is the free vector space w/ basis $V\times W$. An arbitrary element of $F^{(V,W)}$ is a formal linear combination

$$x = \sum_{(v,w)\in V\times W} \alpha_{(v,w)}\,(v,w)$$

of elements of $V\times W$ w/ finitely many nonzero coefficients. Passing to the quotient gives an arbitrary element of $V\otimes W$:

$$\begin{aligned} x + N &= \sum \alpha_{(v,w)}(v,w) + N \\ &= \sum \left(\alpha_{(v,w)}(v,w) + N\right) \\ &= \sum \left((\alpha_{(v,w)}v, w) + N\right) \qquad \text{replace } \alpha_{(v,w)}v \text{ by } v' \\ &= \sum v'\otimes w. \end{aligned}$$

So: an arbitrary element of $V\otimes W$ is just a finite sum of simple tensors

$$\sum_{i=1}^{n} v_i\otimes w_i, \qquad \text{some } n\in\mathbb{N},\ v_i\in V,\ w_i\in W.$$

In particular, the set $\{v\otimes w \mid v\in W, w\in W\}$ is a spanning set for $V\otimes W$.

3.3.4 Theorem (Universal Property of $\otimes$)

Let V, W be vector spaces over $\mathbb{F}$. For any vector space Z and any bilinear map $B : V \times W \to Z$, there is a unique linear map $L : V \otimes W \to Z$ with the property that $L(v \otimes w) = B(v, w)$ for all $v \in V$ and $w \in W$. In other words, L is the unique linear map making the diagram

$$\begin{tikzcd}
V \times W \arrow[r, "B"] \arrow[d, "- \otimes -"'] \arrow[dr, phantom, "\circlearrowleft"] & Z \\
V \otimes W \arrow[ur, dashed, "\exists ! L"'] &
\end{tikzcd}$$

commute.

Proof. (existence) We define $L : V \otimes W \to Z$. First, use the Universal Property of the free vector space $F^{(V \times W)}$: since $V \times W$ is a basis for $F^{(V \times W)}$, there is a unique linear map $K : F^{(V \times W)} \to Z$ such that the following diagram commutes

$$\begin{tikzcd}
V \times W \arrow[r, "B"] \arrow[d, "i"'] \arrow[dr, phantom, "\circlearrowleft"] & Z \\
F^{(V \times W)} \arrow[ur, dashed, "\exists ! K"'] &
\end{tikzcd}$$

.

Now, invoke the Universal Property of the Quotient space $F^{(V \times W)}/N$: since B is bilinear, K annihilates

N, i.e. $N \subset \ker(K)$. There exists a unique linear map $L : V \otimes W \to Z$ making the following diagram commute

$$\begin{tikzcd}
F^{(V\times W)} \arrow[r, "K"] \arrow[d, "\pi"'] \arrow[dr, phantom, "G"] & Z \\
V \otimes V \arrow[ur, dashed, "\exists! L"'] &
\end{tikzcd}$$

Together we get the following commutative diagram

$$\begin{tikzcd}
V \times W \arrow[r, "B"] \arrow[d, "- \otimes - \; = \; \pi \circ i"'] \arrow[dr, phantom, "G"] & Z \\
V \otimes W \arrow[ur, "L"'] &
\end{tikzcd}$$

So existence is proved.

(uniqueness) Suppose L' is another linear map satisfying $L'(-\otimes -) = B$. Then for any simple tensor $v \otimes w$,

$$L'(v \otimes w) = B(v,w) = L(v \otimes w).$$

Thus, $L' = L$. $\square$

Remark: The Universal Property of the Tensor Product establishes a bijection: for any vector space Z,

$$\begin{aligned}
\mathrm{Bil}(V \times W, Z) &\xrightarrow{\sim} \mathrm{Hom}(V \otimes W, Z) \\
B &\longmapsto L \\
L(-\otimes -) &\longleftarrow\!\shortmid \; L
\end{aligned}$$

Thus, there is a one-to-one correspondance between bilinear maps from $V \times W$ to Z and linear maps from $V \otimes W$ to Z.

Remark: For vector spaces $V_1, \ldots, V_n$, one can define

$$V_1 \otimes \cdots \otimes V_n$$

either recursively $V_1 \otimes \cdots \otimes V_n := (((V_1 \otimes V_2) \otimes V_3) \cdots) \otimes V_n)$ or by a suitable quotion $F^{(V_1 \times \cdots \times V_n)}/N$. One can show that both constructions are isomorphic and that the tensor product construction is associative so that the recursive construction is independent of the bracketing. It has a Universal Property which establishes a bijection

$$\mathrm{Mult}(V_1 \times \cdots \times V_n, Z) \xrightarrow{\sim} \mathrm{Hom}(V_1 \otimes \cdots \otimes V_n, Z)$$

for any vector space Z.