2.8 Direct Sums and Quotients

2.8.1 Definition

Let $V_1, \dots, V_n$ be vector spaces over F. Their (external) direct sum is the set

$$\bigoplus_{i=1}^{n} V_i := \left\{ (v_1, \dots, v_n) \;\middle|\; v_i \in V_i,\ i=1,\dots,n \right\}$$

together with the operations

(addition) $(v_1, \dots, v_n) + (w_1, \dots, w_n) := (v_1+w_1, \dots, v_n+w_n)$

(scalar mult) $\alpha(v_1, \dots, v_n) := (\alpha v_1, \dots, \alpha v_n)$.

□

Remark: $\bigoplus_{i=1}^{n} V_i$ is a vector space over F. As a set $\bigoplus_{i=1}^{n} V_i$ is just the cartesian product of sets $\prod_{i=1}^{n} V_i$. There is good reason to use "$\oplus$" instead of "$\prod$".

Remark: One can define direct sum for an arbitrary family V_i, $i \in I$ of vector spaces. One starts by defining the direct product

$$\prod_{i \in I} V_i := \left\{ f : I \longrightarrow \bigcup_{i \in I} V_i \;\middle|\; f(i) \in V_i \ \forall i \in I \right\}.$$

This is a vector space with "coordinate-wise" addition and scalar mult:

$$(f+g)(i) := f(i) + g(i) \in V_i$$

$$(\alpha f)(i) := \alpha f(i).$$

The direct sum is the subspace consisting of functions for which $f(i) \neq 0$ for finitely many $i \in I$

$$\bigoplus_{i\in I} V_i := \left\{ f \in \prod_{i \in I} V_i \;\middle|\; \begin{array}{l} f(i) = 0 \text{ for all but} \\ \text{finitely many } i \in I \end{array} \right\}.$$

It is customary to write the symbol $\sum_{i\in I} f_i$ $(f_i = f(i))$ to represent the function $f \in \bigoplus_{i\in I} V_i$.

Then

$$\sum_{i\in I} f_i + \sum_{i\in I} g_i = \sum_{i\in I} f_i + g_i$$

$$\alpha \sum_{i\in I} f_i = \sum_{i\in I} \alpha f_i \qquad .$$

When the index set I is finite $\prod_{i\in I} V_i = \bigoplus_{i\in I} V_i$

2.8.2 Example

Let F be a field. Then

$$F^n = \{ (a_1, \dots, a_n) \mid a_i \in F \}$$

$$= \underbrace{F \oplus \cdots \oplus F}_{n\text{-summands}}$$

More generally, let X be any set. A long time ago we defined a subspace $F^{(X)}$ of F^X

$$F^{(X)} = \left\{ f : X \to F \;\middle|\; \begin{array}{l} f(x) = 0 \text{ for all but} \\ \text{finitely many } x \in X \end{array} \right\}$$

This is called the free vector space w/ basis X. Since $F = \bigcup_{x \in X} F$, we see that $F^X = \prod_{x \in X} F$ and $F^{(X)} = \bigoplus_{x \in X} F$.

$\square$

2.8.3 Proposition The set $\{(0,0,\dots,v,\dots,0) \mid v \in V_i\}$ is a subspace of $\bigoplus_{i=1}^{n} V_i$ that is isomorphic to V_i.

Proof. Clearly, the set is a subspace. Define $c_i : V_i \longrightarrow \bigoplus_{i=1}^{n} V_i$ via $c_i(v) = (0, \dots, \underset{\uparrow\, i^{th} \text{ slot}}{v}, \dots, 0)$. Clearly c_i is injective and the image of c_i is $\{(0,\dots,v,\dots,0) \mid v \in V_i\}$.

$\square$

Remark: For this reason, we usually identify V_i with the subspace $\{(0,0,\dots,v,\dots,0) \mid v \in V_i\}$ of $\bigoplus_{i=1}^{n} V_i$. In this way, one can think of V_i as being genuine subspace of $\bigoplus_{i=1}^{n} V_i$. The subspaces $V_1, \dots, V_n$ of $\bigoplus_{i=1}^{n} V_i$ are special:

(a) $\bigoplus_{i=1}^{n} V_i = \sum_{i=1}^{n} V_i$

(b) For each i, $V_i \cap \left(\sum_{j \neq i} V_j \right) = \{0\}$.

2.8.4 Definition Let V be a vector space w/ subspaces $V_1, \dots, V_n$. We say that V is the **(internal) direct sum** of $V_1, \dots, V_n$ if

(a) $V = \sum_{i=1}^{n} V_i$

(b) For all $i = 1, \dots, n$, $V_i \cap \left(\sum_{j \neq i} V_j \right) = \{0\}$.

□

2.8.5 Proposition Let V be a vector space and $V_1, \dots, V_n$ subspaces. The following are equivalent.

(a) V is the internal direct sum of $V_1, \dots, V_n$

(b) For each $v \in V$, there are unique vectors $v_1, \dots, v_n$ w/ $v_i \in V_i$, $i = 1, \dots, n$ such that $v = \sum_{i=1}^{n} v_i$.

(c) There is an isomorphism $\varphi : V \longrightarrow \bigoplus_{i=1}^{n} V_i$ (from V to the external direct sum) sending each subspace $V_i \subseteq V$ to $\{(0, \dots, v, \dots, 0) \mid v \in V\} \subseteq \bigoplus_{i=1}^{n} V_i$.

Proof. ((c) ⇒ (a)) from the remarks above.

((a) → (b)) Suppose V is the internal direct sum of $V_1, \dots, V_n$. Let $v \in V$. By def, there exists $v_i \in V_i$ such that $v = \sum_{i=1}^{n} v_i$. Suppose additionally that $v = \sum_{i=1}^{n} w_i$. Then for each i

$$V_i \ni v_i - w_i = \sum_{j \neq i} w_j - \sum_{j \neq i} v_j = \sum_{j \neq i} w_j - v_j \in \sum_{j \neq i} V_j$$

Thus, $v_i - w_i \in V_i \cap \left(\sum_{j\neq i} V_j\right) = \{0\}$. Thus, $v_i = w_i$.

((b)⇒(c)) Define $\varphi : V \longrightarrow \hat{\bigoplus}_{i=1}^{n} V_i$ as follows. Given $v \in V$, there are unique $v_i \in V_i$ such that $v = \sum_{i=1}^{n} v_i$. So define $\varphi(v) = (v_1, \dots, v_n)$. By uniqueness, this is a function. An inverse $\psi : \bigoplus V_i \longrightarrow V$ is given by $\psi((v_1, \dots, v_n)) = \sum_{i=1}^{n} v_i$. Clearly, φ is linear, hence an isomorphism. Further,

$$\varphi(V_i) = \{(0, \dots, v, \dots, 0) \mid v \in V_i\}.$$

☐

Notation: Because of Prop 2.8.4, we write $V = \hat{\bigoplus}_{i=1}^{n} V_i$ if V is the internal direct sum of subspaces $V_1, \dots, V_n$.

2.8.5 Proposition Let $V_1, \dots, V_n$ be subspaces of V. Then $V = \hat{\bigoplus}_{i=1}^{n} V_i$ if and only if there are bases $B_1, \dots, B_n$ for $V_1, \dots, V_n$ such that $B = \hat{\bigcup}_{i=1}^{n} B_i$ is a basis for V.

Proof. Exercise. ☐

2.8.6 Corollary If $V_1, \dots, V_n$ are vector spaces over $\mathbb{F}$, then

$$\dim_{\mathbb{F}}\left(\bigoplus_{i=1}^{n} V_i\right) = \sum_{i=1}^{n} \dim_{\mathbb{F}}(V_i).$$

☐

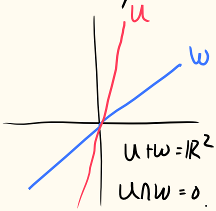
2.8.7 Definition Let W be a subspace of V. A complement of W is a subspace U of V such that $V = W \oplus U$. $\square$

Remark: Every subspace has a complement. Let $\{w_1, \dots, w_n\}$ be a basis for $W \subseteq V$. Extend it to a basis $\{w_1, \dots, w_n, w_{n+1}, \dots, w_m\}$ for V. Then $\mathcal{U} := \text{span}\{w_{n+1}, \dots, w_m\}$ is a complement of W. If V is infinite-dimensional, one can still prove the claim using the axiom of choice. $\square$

2.8.8 Example A subspace typically has many complements

- $V = \mathbb{R}^2$, W = any 1-dim subspace (line thru the origin)

  Any other 1-dim subspace is a complement of W.



A vector space typically has many different direct sum decompositions

- if $\{b_1, \dots, b_n\}$ is a basis for V, then

$$V = \text{span}(b_1) \oplus \cdots \oplus \text{span}(b_n).$$

$\square$

2.8.9 Definition Let V be a vector space and let W be a subspace. We define a relation $\sim_W$ on V as follows:

$$u \sim_W v \text{ if and only if } u - v \in W.$$

$\square$

2.8.10 Proposition Let V be a vector space and let W be a subspace. Then

(a) $\sim_W$ is an equivalence relation

(b) The equivalence class of $v \in V$ is the set

$$v + W := \{v + w \mid w \in W\}.$$

Proof. (a) reflexive: if $x \in V$, then $x - x = 0 \in W$.

So $x \sim_W x$.

Symmetric: if $x \sim_W y$, then $x - y \in W$.

$\Rightarrow y - x = -(x-y) \in W$

$\Rightarrow y \sim_W x$.

transitive: if $x \sim_W y$ and $y \sim_W z$, then $x - y,\ y - z \in W$.

$\Rightarrow x - z = (x-y) + (y-z) \in W$.

$\Rightarrow x \sim_W z$.

(b) Write $[v]$ for the equivalence class of $v \in V$.

Then $[v] = \{u \in V \mid u \sim_W v\}$

$= \{u \in V \mid u - v \in W\}$

$= \{v + w \mid w \in W\} = v + W$. $\square$

2.8.11 Definition Let V be a vector space and W a subspace. The equivalence classes $v+W$ are called cosets of V w.r.t. W. The quotient of V by W is the set V/W of all cosets

$$V/W := \{v+W \mid v \in V\}$$

together w/ the operations of addition and scalar multiplication defined as follows:

(addition) $(u+W)+(v+W) := (u+v)+W$

(scalar mult) $\alpha(u+W) := (\alpha u)+W.$

□

2.8.12 Proposition The operations of addition and scalar multiplication on V/W are well-defined, making V/W into a vector space.

Proof. We must show that the operations are well-defined, i.e. they do not depend on the choice of coset reps.

Suppose $u+W = u'+W$ and $v+W = v'+W$. Then we have $\to \boxed{u-u' \in W}$ $\to v-v' \in W$

$$\begin{aligned} u+W+v+W &= u+v+W \\ &= u+v+(u'-u)+(v'-v)+W \\ &= u'+v'+W \\ &= u'+W+v'+W. \end{aligned}$$

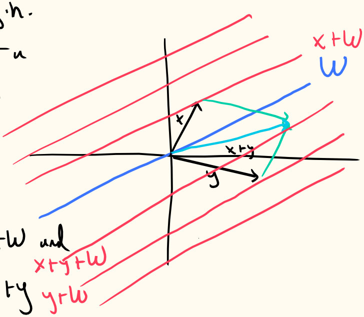
Let $\alpha \in F$. Then

$$\alpha(u+W) = \alpha u + W$$

$$= \alpha u + \alpha(u'-u) + W$$
$$= \alpha u' + W.$$

Easy to check that V/W is a vector space. The zero vector is $W := 0 + W$ and the additive inverse of $u+W$ is $-(u+W) := -u + W$. □

2.8.13 Example Let $V = \mathbb{R}^2$ and W = any 1-dim subspace = any line thru the origin. Let $x \in \mathbb{R}^2$. Then $x + W$ is just a translation of W by x. So $x+W$ is a line parallel to W. So V/W = set of all lines parallel to W. Two lines $x+W$ and $y+W$ are added by computing $x+y$ and then translate W by $x+y$. □

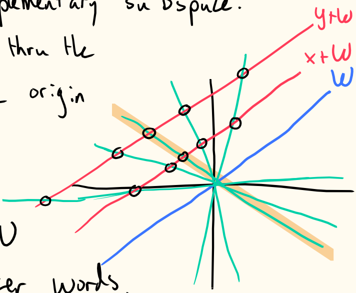


Remark: The quotient space V/W comes equipped w/ an important linear map

$$\begin{aligned} \pi : V &\longrightarrow V/W \\ v &\longmapsto v + W \end{aligned}$$

called the projection map or quotient map. Notice: π is surjective and $\ker \pi = W$. In the above example, the map $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/W$ collapses each line parallel to W to a single point (or vector).

Some Motivation: Given a subspace W of V, there is no canonical choice of complementary subspace. For example, $V=\mathbb{R}^2$, W = line thru the origin. Any other line thru the origin is a complement of W.



Observe: any line parallel to W intersects every complement of W at a single point. In other words, given any complement U of W, we can think of U as a set of coset representatives for V/W: U contains precisely one element from each coset.

2.8.14 Proposition Let U be any complement of W in V. The composition of linear maps

$$U \overset{\subseteq}{\hookrightarrow} V \overset{\pi}{\twoheadrightarrow} V/W$$

is an isomorphism of U onto V/W. In particular,

$$V \cong W \oplus V/W.$$

Proof. By assume, $V = W \oplus U$. Denote the inclusion map $U \hookrightarrow V$ by i. We just need to show that $\pi \circ i$ is bijective.

(surjective) Let $v \in V$. Choose unique elements $w \in W$ and $u \in U$ s.t. $v = w + u$. Then

$$\pi \circ i(u) = \pi(u) = u + W = u + w + W$$
$$= w + u + W$$

$= V + W.$

So $\pi \circ i$ is surjective.

(injective). Suppose $u + W = \pi \circ i(u) = \pi \circ i(u') = u' + W$. Then $u - u' \in W \cap U = \{0\}$. Hence $u - u' = 0$. So $\pi \circ i$ is injective.

Finally, $V = W \oplus U \cong W \oplus V/W$.

□

Remark: By Prop 2.8.14, one can think of V/W as a "canonical complement" of W in V, except that it is not a subspace of V at all.

2.8.5 Theorem (Universal Property of the Quotient)

Let W be a subspace of a vector space V. For every vector space Z and every linear map $L : V \to Z$ satisfying $W \subseteq \ker L$, there exists a unique linear map $\tilde{L} : V/W \to Z$ such that $\tilde{L} \circ \pi = L$. In other words, there is a unique linear map $\tilde{L}$ making the following diagram commute:

$$\begin{tikzcd}
v \arrow[d, mapsto] & V \arrow[r, "L"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & Z \\
v+W & V/W \arrow[ur, dashed, "\exists ! \tilde{L}"'] &
\end{tikzcd}$$

$\left[\begin{array}{l} \tilde{L} \circ \pi = L \\ \tilde{L}(v+W) = L(v) \end{array}\right]$

Proof.

Define $\tilde{L} : V/W \to Z$ via $\tilde{L}(v + W) = L(v)$. Then clearly $\tilde{L} \circ \pi = L$. The question is whether or not $\tilde{L}$ is well-defined.

(well-defined) Assume $v+W = v'+W$. Then $v-v' \in W$. But $W \subseteq \ker L$, hence $L(v)-L(v') = L(v-v') = 0$. Hence, $\tilde{L}(v+W) = L(v) = L(v') = \tilde{L}(v'+W)$.

(linearity) $\tilde{L}(v+W + v'+W) = \tilde{L}(v+v'+W)$

$$= L(v+v')$$

$$= L(v)+L(v')$$

$$= \tilde{L}(v+W) + \tilde{L}(v'+W).$$

Scalar mult is similar.

(uniqueness) Suppose $\bar{L} : V/W \to Z$ is another linear map s.t. $\bar{L} \circ \pi = L$. Then for any $v \in V$,

$$\bar{L}(v+W) = \bar{L}\circ\pi(v) = L(v) = \tilde{L}\circ\pi(v) = \tilde{L}(v+W).$$

So $\bar{L} = \tilde{L}$.

□

Remark: The quotient map $\pi : V/W \to Z$ has the following property:

$$W \subseteq \ker \pi \qquad (\pi \text{ "annilates" } W).$$

The theorem states that π is the "universal" linear map w/ this property in the sense that any other linear map $L : V \to Z$ w/ this property factors uniquely through π:

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & Z \\ V/W \arrow[ur, dotted, "\tilde{L}"'] & \end{tikzcd} \qquad \tilde{L}\circ\pi = L$$

Using the language of category theory, one can give a precise definition of "universal property".

□

Remark: The best way to think about the Universal Prop. of the Quotient is as an answer to the following question:

Question: How do I define a linear map out of V/W.

Answer: Define a linear map out of V which annihilates W.

In other words, there is a bijection

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{F}}(V/W, Z) & \overset{\sim}{\longleftrightarrow} & \{L \in \mathrm{Hom}(V, Z) \mid W \subseteq \ker L\} \\ \tilde{L} & \longmapsto & \tilde{L} \circ \pi \end{array}$$

where Z is any other vector space. Q: is the RHS a vector space? If yes, is the bijection linear?

2.8.16 Corollary (First Isomorphism Theorem)

Let $L: V \to W$ be a linear map. Then there is a canonical isomorphism

$$V/\ker(L) \xrightarrow{\sim} \mathrm{im}(L).$$

More precisely, there is an iso making the following diagram commute:

$$\begin{tikzcd} V \arrow[r, "L"] & W \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[d, "\pi"'] \arrow[r, phantom, "G"] & {} \\ V/\ker L \arrow[r, dashed, "\exists!"] & \mathrm{im}(L) \arrow[u, "\subseteq"'] \end{tikzcd}$$

Proof.

Clearly, L annihilates $\ker L$ by def. So by UPoQ there is a unique linear map $\tilde{L}$ making the diagram commute

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\pi"'] \arrow[dr, phantom, "G"] & W \\ V/\ker(L) \arrow[ur, dashed, "\exists!\tilde{L}"'] & \end{tikzcd}$$

One can prove that $\mathrm{im}(\tilde{L}) = \mathrm{im}(L)$ so get the diagram

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\pi"'] \arrow[dr, phantom, "G"] & W \\ V/\ker(L) \arrow[r, "\tilde{L}"'] & \mathrm{im}(L) \arrow[u, "\subseteq"'] \end{tikzcd}$$

Suffices to show $\tilde{L}$ is injective. But one can show that $\ker \hat{L} = \ker L/\ker L = 0$. $\square$

Remark: Thus, every linear map factors through a quotient map, an isomorphism of spaces of the same dimension, and an inclusion. This is evidence to suggest that quotients are important!

2.8.17 Theorem (Rank-Nullity) Let V be a finite-dimensional vector space and let

$L: V \to W$ be a linear map. Then

$$\dim(V) = \operatorname{rank}(L) + \operatorname{nullity}(L).$$

Proof. Since $\ker L$ is a subspace of V, we have

$$V \cong \ker L \oplus V/\ker L$$

by Prop 2.8.14. Then

$$\begin{aligned}
\dim(V) &= \dim(\ker L \oplus V/\ker L) \\
&= \dim \ker L + \dim V/\ker L \quad (\text{Cor } 2.8.6) \\
&= \dim \ker L + \dim \operatorname{im} L \quad (\text{Cor } 2.8.16) \\
&= \operatorname{nullity}(L) + \operatorname{rank}(L).
\end{aligned}$$

□

Remark: One can prove rank-nullity w/out quotients. One can start w/ a basis for the kernel, extend to a basis for V, and use this to find a basis for the image. Then count dimensions.