2.4 Change of Basis

Recall: An $n \times n$ matrix A is invertible if there exists a (unique) $n\times n$ matrix A^{-1} such that

$$AA^{-1} = I_n = A^{-1}A$$

where $I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ is the $n \times n$ identity.

Recall: Let F be a field. In Prop 2.1.2 we proved that there is a bijection

$$\begin{tikzcd}
\mathrm{End}(F^n) := \left\{\begin{matrix}\text{linear maps}\\ F^n \to F^n\end{matrix}\right\} \arrow[r, leftrightarrow, "\sim"] & \left\{\begin{matrix} n\times n \text{ matrices}\\ \text{over } F\end{matrix}\right\} = F^{n\times n} \\
\rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} \\
\left\{\begin{matrix}\text{isomorphisms}\\ F^n \to F^n\end{matrix}\right\} \arrow[r, leftrightarrow, "\sim"] & \left\{\begin{matrix} n\times n \text{ invertible}\\ \text{matrices}\end{matrix}\right\}
\end{tikzcd}$$

The bijection was given by sending $L: F^n \to F^n$ to the matrix $[L] := (L(e_1)|\cdots|L(e_n))$. The inverse sends a matrix $A \in F^{n\times n}$ to the linear $A\cdot : x \mapsto Ax$.

Observe:

$$\begin{tikzcd}
L: F^n \to F^n \text{ isomorphism} \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & {[L]} \text{ is invertible} \arrow[d, Leftrightarrow] \\
L(e_1), \dots, L(e_n) \text{ basis for } F^n \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & \text{columns of } [L] \text{ basis for } F^n \arrow[d, Leftrightarrow] \\
L(e_1), \dots, L(e_n) \text{ indep. in } F^n \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & \text{columns of } [L] \text{ indep in } F^n \arrow[d, Leftrightarrow] \\
L(e_1), \dots, L(e_n) \text{ span } F^n \arrow[r, Leftrightarrow] & \text{columns of } [L] \text{ span } F^n
\end{tikzcd}$$

Recall: Choosing a basis $B = (b_1, \ldots, b_n)$ for a vector space over F amounts to choosing an isomorphism

$$\begin{tikzcd} V \arrow[r, "\sim"] & F^n \\ v \arrow[r, mapsto] & {[v]_B} \end{tikzcd}$$

Suppose $C = (c_1, \ldots, c_n)$ is another basis for V. Consider the composition

$$\begin{tikzcd} F^n \arrow[r, "\sim"] & V \arrow[r, "\sim"] & F^n \\ {[v]_B} \arrow[r, mapsto] & v \arrow[r, mapsto] & {[v]_C} \\ e_i \arrow[r, mapsto] & b_i \arrow[r, mapsto] & {[b_i]_C} \end{tikzcd}$$

This is an isomorphism from F^n to F^n. Therefore its given by an invertible matrix:

2.4.1 Definition Let $B = (b_1, \ldots, b_n)$ and $C = (c_1, \ldots, c_n)$ be ordered bases for V. The **transition matrix (or change of basis matrix)** from B to C is the $n \times n$ matrix $T^C_B \in F^{n \times n}$ whose i^{th} column is $[b_i]_C$, $1 \leq i \leq n$. I.e.

$$T^C_B = ([b_1]_C \mid \cdots \mid [b_n]_C).$$

□

2.4.2 Proposition Let $B = (b_1, \ldots, b_n)$ and $C = (c_1, \ldots, c_n)$ be ordered bases for V. Then

(a) $T^C_B \in F^{n \times n}$ is the unique $n \times n$ matrix satisfying $T^C_B [v]_B = [v]_C$ for all $v \in V$. In other words

T_B^C is the unique matrix making the following diagram commute:

$$\begin{tikzcd}
v \arrow[r, mapsto] & {[v]_B} \\
V \arrow[r, "\sim"] \arrow[dr, "\sim"'] \arrow[dr, phantom, "\circlearrowleft", near end, shift left=3ex] & F^n \arrow[d, "T_B^C"] \\
 & F^n \\
v \arrow[r, mapsto] & {[v]_C}
\end{tikzcd}$$

(b) T_B^C is invertible with $(T_B^C)^{-1} = T_C^B$

(c) (change of basis is transitive) If $D = (d_1, \dots, d_n)$ is a third basis for V, then $T_B^D = T_C^D T_B^C$.

Proof.

(a) Let $v \in V$. Then $T_B^C [v]_B = [v]_C$ by definition. Suppose $A \in F^{n \times n}$ satisfies $A[v]_B = [v]_C$ for all $v \in V$. Then for each $i \in \{1, \dots, n\}$

$$i^{th} \text{ column of } A = Ae_i = A[b_i]_B = [b_i]_C.$$

Thus, $A = T_B^C$.

(b) We already know T_B^C is invertible because it is the matrix representing an isomorphism. Let $v \in V$. Then using (a)

$$[v]_B = T_C^B [v]_C = T_C^B T_B^C [v]_B$$

$$[v]_C = T_B^C [v]_B = T_B^C T_C^B [v]_C$$

Since $\{[v]_B \mid v \in V\} = \{[v]_C \mid v \in V\} = F^n$, this implies $T_C^B T_B^C = I_n = T_B^C T_C^B$.

(c) Let $v \in V$. Then

$$T_C^D T_B^C [v]_B = T_C^D [v]_C = [v]_D .$$

By part (a), $T_C^D T_B^C = T_B^D$ since T_B^D is the unique matrix w/ that property.

Alternate proof: Meditate upon the diagram.

$$\begin{tikzcd}
& \mathbb{F}^n & \text{D-coord.} \\
& V \arrow[u, "\wr"'] \arrow[dl, "\wr"'] \arrow[dr, "\wr"] & \\
\mathbb{F}^n \arrow[uur, bend left, "T_B^D"] \arrow[rr, "T_B^C"'] & & \mathbb{F}^n \arrow[uul, bend right, "T_C^D"'] & \text{C-coord} \\
\text{B-coord} & &
\end{tikzcd}$$

☐

Note: part (c) is useful for computing T_B^C if V has a "standard basis" E. If E is standard, then T_B^E and $(T_C^E)^{-1}$ are easy to compute and $T_B^C = (T_C^E)^{-1} T_B^E$.

2.4.3 Example

$E = ((1,0),(0,1))$, $B = ((1,2),(3,4))$, $C = ((-1,1),(1,2))$ are bases for $\mathbb{R}^2$. Compute T_B^C.

$$T_B^E = \begin{pmatrix} [(1,2)]_E & [(3,4)]_E \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \qquad (T_C^E)^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Then $T_B^C = \frac{1}{3}\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 & -2 \\ 3 & 7 \end{pmatrix}$.

☐

2.5 Algebra of Linear Maps

Notation: Let V and W be vector spaces over F.

The set of all linear maps $L: V \longrightarrow W$ is denoted by

$$\mathrm{Hom}_F(V,W) \subseteq \mathrm{Maps}(V,W) = W^V.$$

One writes $\mathrm{End}_F(V) := \mathrm{Hom}_F(V,W)$.

Recall: Since W is a vector space, W^V is also a vector space under the operations

$$(f+g)(v) := f(v) + g(v) \qquad \forall f,g \in W^V$$
$$(\alpha f)(v) := \alpha f(v) \qquad \alpha \in F.$$

2.5.1 Theorem Let V and W be vector spaces over F. Then the set $\mathrm{Hom}_F(V,W)$ of linear maps from V to W is a subspace of the vector space $W^V = \mathrm{Maps}(V,W)$ of all functions from V to W.

Proof. Clearly, $0: V \longrightarrow W$ is linear, so $\mathrm{Hom}(V,W) \neq \emptyset$. Let $L, K \in \mathrm{Hom}(V,W)$ and $\alpha \in F$. It's an easy exercise to check that $L+K$ and αL are also linear maps. Hence, $L+K, \alpha L \in \mathrm{Hom}(V,W)$.

$\square$

When $V = W$, one can even define a binary product on $\mathrm{End}(V)$: for $L, K : V \longrightarrow V$, the product of K and L is the composition

$$KL := K \circ L.$$

We already proved in Prop 2.1.10 that $KL \in \mathrm{End}(V)$.

2.5.2 Proposition Let $\alpha \in F$ and $L, S, T \in \mathrm{End}_F(V)$.

Then

(a) $L(ST) = (LS)T$

(b) $L(S+T) = LS + LT$

(c) $(S+T)L = SL + TL$

(d) $\alpha(ST) = (\alpha S)T = S(\alpha T)$.

Proof. Easy exercise.

□

So $\mathrm{End}_F(V)$ is an F-vector space, but it also has a way to "multiply vectors" and this operation is compatible w/ addition and scalar multiplication. Such an algebraic structure is called an algebra over F.

You already know several algebras:

2.5.3 Example (algebras)

(a) $\mathrm{End}_F(V)$ over F.

(b) $F^{n \times n}$ over F

(c) F over F

(d) $\mathbb{C}$ over $\mathbb{Q}$

(e) $F[x]$ over F (Note: $F_n[x]$ is not an algebra since degree increases when you multiply polynomials.)

□

2.6 Matrix Representations of Linear Maps

We have already seen that every linear map $F^n \to F^m$ is given by left mult by a matrix $A \in F^{m\times n}$. Suppose V and W are vector spaces w/ bases $B = (b_1, \dots, b_n)$ and $C = (c_1, \dots, c_m)$ and suppose $L : V \to W$ is a linear map. We have isomorphisms $V \cong F^n$ and $W \cong F^m$.

$$\begin{tikzcd}
v \arrow[r, mapsto] & {[v]_B} \\
V \arrow[r, "\sim"] \arrow[d, "L"'] & F^n \arrow[d, "A\cdot"] \\
W \arrow[r, "\sim"] & F^n \\
w \arrow[r, mapsto] & {[w]_C}
\end{tikzcd}$$

We can complete the diagram by taking the red arrow to be the composition

$$\begin{tikzcd}
e_i \arrow[r, mapsto] & b_i \arrow[r, mapsto] & L(b_i) \arrow[r, mapsto] & {[L(b_i)]_C} \\
F^n \arrow[r, "\sim"] \arrow[rrr, bend right, "A\cdot"'] & V \arrow[r, "L"] & W \arrow[r, "\sim"] & F^m
\end{tikzcd}$$

This map can be represented by a matrix $A \in F^{m\times n}$. Clearly, A will depend on B and C.

2.6.1 Definition The **matrix of a linear map** $L : V \to W$ with respect to the bases $B = (b_1, \dots, b_n)$ and $C = (c_1, \dots, c_m)$ for V and W, respectively is the $m \times n$ matrix $[L]_B^C$ whose i^{th} column is $[L(b_i)]_C$, $1 \leq i \leq n$. I.e.

$$[L]^C_B := \left([L(b_1)]_C \,\middle|\, \cdots \,\middle|\, [L(b_n)]\right) \in F^{m\times n}$$

We write $[L]_B := [L]^B_B$. $\square$

2.6.2 Theorem Let V and W be vector spaces w/ bases $B=(b_1,\dots,b_n)$ and $C=(c_1,\dots,c_m)$.

(a) For each linear map $L:V\to W$, the matrix $[L]^C_B \in F^{m\times n}$ the unique matrix satisfying $[L]^C_B[v]_B=[L(v)]_C$ for all $v\in V$. In other words, $[L]^C_B$ is the unique matrix making the diagram commute:

$$\begin{tikzcd}
V \arrow[r, "\sim"] \arrow[d, "L"'] \arrow[dr, phantom, "\circlearrowleft"] & F^n \arrow[d, "{[L]^C_B}"] \\
W \arrow[r, "\sim"'] & F^n
\end{tikzcd}.$$

(b) The function $\mathrm{Hom}_F(V,W)\longrightarrow F^{m\times n}$ sending a linear map $L:V\to W$ to the matrix $[L]^C_B$ is a vector space isomorphism.

Proof. (a) $[L]^C_B$ has the desired property by definition. Suppose $A\in F^{m\times n}$ has the same property. Let $i\in\{1,\dots,n\}$. Then

$$i^{th}\text{-column of } A = Ae_i = A[b_i]_B = [L(b_i)]_C.$$

(b) The function $L\longmapsto [L]^C_B$ is linear. Let $L,K\in \mathrm{Hom}(V,W)$. Then

$$\begin{aligned}
[L+K]^C_B &= \left([(L+K)(b_1)]_C \cdots\cdots [(L+K)(b_n)]_C\right)\\
&= \left([L(b_1)+K(b_1)]_C \cdots\cdots [L(b_n)+K(b_n)]_C\right)
\end{aligned}$$

$$= \left([L(b_1)]_C + [K(b_1)]_C \cdots [L(b_n)]_C + [K(b_n)]_C \right)$$

$$= \left([L(b_1)]_C \cdots [L(b_n)]_C \right) + \left([K(b_1)]_C \cdots [K(b_n)]_C \right)$$

$$= [L]^C_B + [K]^C_B .$$

Similarly, $[\alpha L]^C_B = \alpha [L]^C_B$ for any $\alpha \in F$.
An inverse $F^{m \times n} \longrightarrow \mathrm{Hom}(V, W)$ is given by sending $A \in F^{m\times n}$ to the linear map

$$V \xrightarrow{\sim} F^n \xrightarrow{A\cdot} F^m \xrightarrow{\sim} W .$$

□

2.6.3 Corollary If V and W are finite-dimensional vector spaces, then so is $\mathrm{Hom}(V, W)$ and $\dim_F \mathrm{Hom}(V, W) = \dim_F V \cdot \dim_F W$.

Proof. The dimension $F^{m \times n}$ is $m \cdot n$.

□

The next result could be useful for calculating the matrix of a linear map if one space has a std. basis.

2.6.4 Proposition Let V be a finite-dimensional vector space w/ bases A, B and let W be a finite-dimensional vector space w/ bases C and D. If $L : V \longrightarrow W$ is a linear map, then

$$T^D_C [L]^C_A = [L]^D_B T^B_A .$$

Proof. One can visualize the claim w/ a diagram:

$$\begin{tikzcd}
\text{A coord} \arrow[r] & F^n \arrow[rrrr, "{[L]_A^C}"] \arrow[ddr, "\sim"] \arrow[dddd, "T_A^B"'] & & & & F^m \arrow[dddd, "T_C^D"] & \text{C coord} \arrow[l] \\
 & & & G & & & 4 \\
 & G & V \arrow[rr, "L"] \arrow[ddl, "\sim"] & & W \arrow[uur, "\sim"] \arrow[ddr, "\sim"] & G & \\
 & & & G & ③ \quad ② & & \\
\text{B coord} \arrow[r] & F^n \arrow[rrrr, "{[L]_B^D}"'] & & & & F^m & \text{D coord} \arrow[l] \\
 & & & & ① & &
\end{tikzcd}$$

The claim is that the blue outer square commutes. To find the proof, use commuting triangles and squares. 1 = 2 = 3 = 4 = 5. This proves the claim. This argument is called a diagram chase.

Alternate proof:

$$\begin{aligned}
[L]_B^D \, T_A^B [v]_A = [L]_B^D [v]_B &= [L(v)]_D \\
&= T_C^D [L(v)]_C \\
&= T_C^D [L]_A^C [v]_A . \qquad \square
\end{aligned}$$

2.6.5 Example For any field F, formal diff. of polynomials defines a linear map $D : F_n[x] \to F_n[x]$.

(a) Compute $[D]_E$ where $E = (1, x, \dots, x^n)$.

$$[D]_E = \left([0]_E \; [1]_E \; [2x]_E \; [3x^2]_E \; \cdots \; [nx^{n-1}]_E \right)$$

$$= \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 2 & 0 & & 0 \\
0 & 0 & 0 & 3 & & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 & & & & & n \\
0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}$$

(b) Consider $n=2$, $F=\mathbb{Z}_2$, $B=(1+x, x+x^2, x)$, $C=(1, x+x^2, 1+x^2)$. Compute $[D]^C_B$ using Prop 2.6.4: $[D]^C_B = (T^E_C)^{-1}[D]^E_B$.

$2=0$

$$
\begin{aligned}
[D]^E_B &= \left([D(1+x)]_E \; [D(x+x^2)]_E \; [D(x)]_E \right) \\
&= \left([1]_E \; [1]_E \; [1]_E \right) \\
&= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
(T^E_C)^{-1} &= \left([1]_E \; [x+x^2]_E \; [1+x^2]_E \right)^{-1} \\
&= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}
\end{aligned}
$$

So $$[D]^C_B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

□

2.6.6 Proposition Suppose $L: U \to V$ and $K: V \to W$ are linear maps between finite-dimensional vector spaces U, V, W with bases B, C, D, respectively. Then

$$[KL]^D_B = [K]^D_C [L]^C_B.$$

Proof. Diagram chase

$$\begin{tikzcd}
U \arrow[r, "L"] \arrow[d, "\wr"'] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[r, "K"] \arrow[d, "\wr"'] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "\wr"'] \\
F^n \arrow[r, "{[L]^C_B}"'] & F^m \arrow[r, "{[K]^D_C}"'] & F^k
\end{tikzcd}$$

Both small squares commute by 2.6.2(a)

There for the outer square

$$\begin{tikzcd}
U \arrow[r, "KL"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d] \\
F^n \arrow[r, "{[K]^D_C[L]^C_B}"'] & F^k
\end{tikzcd}$$

But by Thm 2.6.2(a), the unique matrix making the blue square commute is the matrix $[KL]^D_B$. Therefore $[KL]^D_B = [K]^D_C [L]^C_B$ □

2.6.7 Corollary Let V be a finite-dimensional vector space w/ basis B. The function

$$\begin{aligned} \mathrm{End}_F(V) &\longrightarrow F^{n\times n} \\ L &\longmapsto [L]_B \end{aligned}$$

defines an isomorphism of F-algebras.

Proof. We already know that the map is a vector space isomorphism. To be an algebra isomorphism, it just needs to preserve the product. But we just proved that $[KL]_B = [K]_B [L]_B$. □

2.7 Similarity of Linear Operators

Recall: A linear map $L: V \to V$ is called a linear operator. The set of all linear operators is denoted

$\mathrm{End}(V) = \mathrm{Hom}(V,V)$. We proved that every linear operator can be represented by a matrix $[L]_{\mathcal{B}}$ by choosing a basis $\mathcal{B}$ for V. Different bases produce different matrices.

Question: How can we understand the many different matrix representations of an operator L?

Answer: Similarity of matrices.

2.7.1 Definition Two matrices $X, Y \in F^{n\times n}$ are called **similar** if there exists an invertible matrix $P \in F^{n \times n}$ such that $Y = P^{-1}XP$. We write $X \sim Y$ if X and Y are similar. $\square$

Similarity defines an equivalence relation on the set $F^{n\times n}$.

2.7.2 Theorem Let V be an F-vector space of dimension n. Two matrices $X, Y \in F^{n\times n}$ are similar if and only if they represent the same linear operator on V (with respect to possibly different bases for V).

Proof. ($\Rightarrow$) Suppose $X \sim Y$. Then theres an invertible matrix P such that $Y = P^{-1}XP$. We need to find a linear operator $L : V \to V$ and bases $\mathcal{B}, \mathcal{C}$ for V such that $X = [L]_{\mathcal{B}}$ and $Y = [L]_{\mathcal{C}}$. Let

B be any basis for V. Using the isomorphism $\mathrm{End}(V) \xrightarrow{\sim} F^{n \times n}$, choose a linear operator L such that $X = [L]_B$. Since P is invertible, we have an isomorphism

$$\varphi : F^n \xrightarrow{\overset{P\cdot}{\sim}} F^n \xrightarrow{\overset{e_i \mapsto b_i}{\sim}} V$$

So the images of $(e_1, \dots, e_n)$ under φ forms a basis $C = (\varphi(e_1), \dots, \varphi(e_n))$ for B. We claim $Y = [L]_C$. Consider the following diagram

$XP = PY$
$\iff$
$Y = P^{-1}XP$

$$\begin{tikzcd}
F^n \arrow[r, "\overset{P\cdot}{\sim}"] \arrow[d, "Y\cdot"'] \arrow[dr, phantom, "\circlearrowright"] & F^n \arrow[r, "\overset{e_i \mapsto b_i}{\sim}"] \arrow[d, "X\cdot"] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, "L"] \\
F^n \arrow[r, "P\cdot"'] & F^n \arrow[r] & V
\end{tikzcd}$$

$X = [L]_B$
+ Thm 2.6.2

Therefore the outer square

$$\begin{tikzcd}
F^n \arrow[r, "\varphi"] \arrow[d, "Y\cdot"'] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, "L"] \\
F^n \arrow[r, "\varphi"'] & V
\end{tikzcd}$$

commutes. But the unique matrix making the square commute is $[L]_C$. So $Y = [L]_C$ by Thm 2.6.2.

($\Leftarrow$) Suppose X and Y represent the same linear operator L in bases B and C. Then

$$Y = [L]_C = T^C_B [L]_B T^B_C = (T^B_C)^{-1} X T^B_C .$$

Thus, $X \sim Y$. $\square$