1.4.19 Example Let F be a field and X a set.

$$F^X := \mathrm{Maps}(X, F) = \{ f : X \to F : f \text{ a function} \}$$

is an F-vector space w/ "coordinate-wise" addition & scalar mult.

- $X = \{1, \ldots, n\}$, a function $f : \{1, \ldots, n\} \to F$ is just an ordered tuple $(f(1), \ldots, f(n)) \in F^n$

So $F^{\{1, \ldots, n\}} = F^n$

- $X = \mathbb{N}$, a function $f : \mathbb{N} \to F$ is by definition a sequence $(f(1), f(2), \ldots)$

So $F^{\mathbb{N}}$ is the space of of infinite seq. in F.

There is an important subspace of F^X:

$$F^{(X)} := \left\{ f : X \to F \;\middle|\; \begin{array}{l} f(x) = 0 \text{ for all but finitely} \\ \text{many elmts of } X \end{array} \right\}$$

The vector space $F^{(X)}$ has an obvious basis: for each $x \in X$, define $e_x : X \to F$ via

$$e_x(y) = \begin{cases} 1, & y = x \\ 0, & y \neq x. \end{cases}$$

Then clearly $\{e_x : x \in X\}$ is a basis for $F^{(X)}$. However, there is no general way to write down a basis for F^X (if X is infinite). Even though Thm 1.4.18 guarantees the existence of basis, the proof is nonconstructive - it doesnt tell us how to build a basis. In order understand infinite-dimensional vector spaces, one needs new ideas. □

1.4.20 Definition (free vector space) Let F be a field and let X be a set. The vector space $F^{(X)}$ described in Example 1.4.19 is called the free F-vector space with basis X. □

Note: one reason why we care: for any set X, there is a vector space w/ basis (in bijection w/) X.

Chapter II: Linear Maps

2.1 Linear Maps

2.1.1 Definition Let V and W be vector spaces over the same field F. A linear map from V to W is a function $L: V \to W$ satisfying:

(a) $L(u+v) = L(u) + L(v)$ for all $u, v \in V$

(b) $L(\alpha u) = \alpha L(u)$ for all $u \in V$, $\alpha \in F$.

If $V = W$, then L is called a linear operator.

Linear maps are also called linear transformation.

$\square$

Note:
- $L(0) = 0$ $\quad (L(0) = L(0) + L(0) \Rightarrow L(0) = 0)$
- $L(-v) = -L(v)$ $\quad (L(-v) = L((-1)v) = (-1)L(v) = -L(v))$
- $L\left(\sum_{i=1}^{n} \alpha_i v_i\right) = \sum_{i=1}^{n} \alpha_i L(v_i)$ (Proof by induction).

2.1.2 Example

(a) The zero map
$$\begin{aligned} 0 : V &\longrightarrow W \\ v &\longmapsto 0 \end{aligned}$$

(b) The identity map
$$\begin{aligned} \mathrm{id}_V : V &\longrightarrow V \\ v &\longmapsto v \end{aligned}$$

(c) Transpose of a matrix:
$$\begin{aligned} (-)^T : F^{m\times n} &\longrightarrow F^{n\times m} \\ A &\longmapsto A^T \end{aligned}$$

Note: this is a bijection since
$$(A^T)^T = A$$

(d)
$$\begin{aligned} F^{n\times 1} &\longrightarrow F^{n} \\ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} &\longmapsto (a_1, \dots, a_n) \end{aligned} \qquad \begin{aligned} F^{1\times n} &\longrightarrow F^{n} \\ (a_1, \dots, a_n) &\longmapsto (a_1, \dots, a_n) \end{aligned}$$

These are both linear bijection. From now on, we identify the spaces of row vectors, column vectors, and tuples.

(e) Let $A \in F^{m\times n}$. Left multiplication by A defines a linear map

$$\begin{aligned} A\cdot : F^{n\times k} &\longrightarrow F^{m\times k} \\ X &\longmapsto AX \end{aligned}$$

In particular, we have a linear map

$$\begin{tikzcd} F^n \arrow[r, "A\cdot"] & F^m \\ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \arrow[r, mapsto] & A\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \end{tikzcd}$$

(f) Let $A \in F^{m\times n}$. Right mult. by A defines a linear map

$$\begin{tikzcd} \cdot A : F^{k\times n} \arrow[r] & F^{k\times n} \\ X \arrow[r, mapsto] & XA \end{tikzcd}$$

The linear maps $A\cdot$ and $\cdot A$ commute w/ $(-)^T$

$$\begin{tikzcd} F^{n\times k} \arrow[r, "A\cdot"] \arrow[d, "(-)^T"'] \arrow[dr, phantom, "\circlearrowleft"] & F^{m\times k} \arrow[d, "(-)^T"] \\ F^{k\times n} \arrow[r, "\cdot A^T"'] & F^{k\times m} \end{tikzcd}$$

The diagram commutes in the sense that $(AX)^T = X^T A^T$ for all $X \in F^{n\times k}$

When $k=1$

column vectors

row vectors

$$\begin{tikzcd} F^n \arrow[r, "A\cdot"] \arrow[r, bend left=40] \arrow[d, "\wr"'] \arrow[d, bend right=60] \arrow[dr, phantom, "\circlearrowleft"] & F^m \arrow[d, "\wr"] \\ F^n \arrow[r, "\cdot A^T"'] \arrow[r, bend right=40] & F^m \arrow[u, bend right=60] \end{tikzcd}$$

So up to viewing elements of F^n as row or column vectors, the maps $A\cdot$ and $\cdot A^T$ are the same. For this reason, there is no loss in generality in considering only $A\cdot$. $\square$

2.1.3 Proposition Let F be a field. There is a bijection

$$\mathrm{Hom}(F^n,F^m) := \left\{ \begin{array}{c} \text{linear maps} \\ F^n \longrightarrow F^m \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} m \times n \text{ matrices} \\ \text{w/ coeff. in} \\ F \end{array} \right\} =: F^{m\times n}$$

Proof. Define $\Phi : \mathrm{Hom}(F^n,F^m) \longrightarrow F^{m\times n}$ as follows:
$\Phi(L)$ is the $m\times n$ matrix w/ i^{th} column equal to $L(e_i)$, that is

$$\Phi(L) = \big(L(e_1) \,|\, L(e_2) \,|\, \cdots \,|\, L(e_n) \big).$$

Define $\Psi : F^{m\times n} \longrightarrow \mathrm{Hom}(F^n,F^m)$ via $\Psi(A) := A\cdot$.
Then Φ and Ψ are mutually inverse bijections:

$$\begin{aligned}
(\Psi \circ \Phi)(L)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} &= \big(L(e_1) \,|\, L(e_2) \,|\, \cdots \,|\, L(e_n) \big) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \\
&= a_1 L(e_1) + \cdots + a_n L(e_n) \\
&= L(a_1 e_1 + \cdots + a_n e_n) \\
&= L\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.
\end{aligned}$$

the columns of A

$$(\Phi \circ \Psi)(A) = \Phi(A\cdot) = \big(Ae_1 \,|\, Ae_2 \,|\, \cdots \,|\, Ae_n \big) = A.$$

$\square$

The preceding prop. gives us a way to generate examples of linear maps.

2.1.4 Example Let $F = \mathbb{R}$

(a) $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\quad \begin{array}{c} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{array}$ $\quad$ zero map

(b) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\quad \begin{array}{c} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \end{pmatrix} \end{array}$ $\quad$ identity map

(c) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} y \\ x \end{pmatrix}$ reflection over the line $y = x$

(d) $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto 2\begin{pmatrix} x \\ y \end{pmatrix}$ dilates by a factor of 2

(e) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$ projection onto x axis

(f) $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ rotation about origin by θ radians.

$\square$

2.1.5 Example

More exotic examples.

(a) Formal differentiation of polynomials:

$$D : F[X] \longrightarrow F[X]$$

$$a_0 + a_1 X + \cdots + a_n X^n = \sum_{i=0}^{n} a_i x^i \longmapsto \sum_{i=1}^{n} a_i \cdot i \, x^{i-1} = a_1 + 2a_2 X + \cdots + n a_n x^{n-1}$$

(b) $C(\mathbb{R}, \mathbb{R})$ = continuous functions $\mathbb{R} \to \mathbb{R}$, a subspace of $\mathbb{R}^{\mathbb{R}}$. Integration over a compact interval $[a, b]$ defines a linear map:

$$\begin{aligned} C(\mathbb{R}, \mathbb{R}) &\longrightarrow \mathbb{R} \\ f &\longmapsto \int_a^b f(x)\,dx \end{aligned}$$

(c) the set of all convergent sequences of real numbers is subspace of $\mathbb{R}^{\mathbb{N}}$. The limit defines a linear map

$$\{\text{convergent sequences}\} \subseteq \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}$$

$$(a_n) \longmapsto \lim a_n.$$

$\square$

2.1.6 Definition Let $L: V \longrightarrow W$ be a linear map.

(a) The **image** of L is the set

$$\operatorname{im}(L) := \{L(v) : v \in V\}$$

(b) The **kernel** of L is the set

$$\ker(L) := \{v \in V : L(v) = 0\}$$

$\square$

Note: $L: V \longrightarrow W$ is injective $\Leftrightarrow$ $\ker L = \{0\}$

⌜($\Rightarrow$) Let $v \in \ker L$. Then $L(v) = 0 = L(0)$. Since L is inj, $v = 0$.

($\Leftarrow$) Assume $L(v) = L(w)$. Then $v - w \in \ker L$ since $L(v-w) = L(v) - L(w) = 0$. So $v - w = 0$. ⌟

2.1.7 Proposition Let $L: V \longrightarrow W$ be a linear map. Then

(a) $\operatorname{im}(L)$ is a subspace of W

(b) $\ker(L)$ is a subspace of V.

Proof. Exercise.

$\square$

2.1.8 Definition Let $L: V \to W$ be a linear map. The **rank** of L is the dimension of $\mathrm{im}(L)$ and the **nullity** of L is the dimension of $\ker(L)$. We use the notation

$$\begin{aligned} \mathrm{rank}(L) &:= \dim_F(\mathrm{im}(L)) \\ \mathrm{nullity}(L) &:= \dim_F(\ker(L)). \end{aligned}$$

$\square$

2.1.9 Definition A bijective linear map is called an **isomorphism**. Two vector spaces V and W are **isomorphic** if there exists an isomorphism $L: V \xrightarrow{\sim} W$. We write $V \cong W$. $\square$

2.1.10 Proposition If $L: V \to W$ is an isomorphism, then so is $L^{-1}: W \to V$.

Proof. Let $w_1, w_2 \in W$. Then

$$\begin{aligned} L^{-1}(w_1 + w_2) &= L^{-1}\big(L(L^{-1}(w_1)) + L(L^{-1}(w_2))\big) \\ &= L^{-1}\big(L(L^{-1}(w_1) + L^{-1}(w_2))\big) \\ &= L^{-1}(w_1) + L^{-1}(w_2). \end{aligned}$$

Scalar multiplication is similar.

$\square$

2.1.11 Proposition If $K: U \to V$ and $L: V \to W$ are linear maps, then so is $L \circ K: U \to W$.

Proof. Exercise.

$\square$

2.1.12 Corollary The relation $\cong$ on the class of F-vector spaces is an equivalence relation.

Proof. Reflexive: id_V is an isomorphism.

Symmetric: Prop 2.1.10

Transitive: Prop 2.1.11

$\square$

2.2 Coordinates

2.2.1 Definition Let V be a finite-dimensional vector space. An ordered basis for V is a tuple $B = (b_1, \dots, b_n)$ consisting of the elements of a basis in a certain order.

$\square$

2.2.2 Proposition Let $B = (b_1, \dots, b_n)$ be an ordered basis for V. For every $v \in V$, there is a unique vector $(\alpha_1, \dots, \alpha_n) \in F^n$ satisfying

$$v = \sum_{i=1}^{n} \alpha_i b_i .$$

Proof. This follows from Prop 1.4.5.

$\square$

2.2.3 Definition Let $B = (b_1, \dots, b_n)$ be an ordered basis for V. The coordinates of $v \in V$

w/ respect to B are the unique scalars $\alpha_1,\dots,\alpha_n \in F$ satisfying $v=\sum_{i=1}^{n} \alpha_i b_i$. The coordinate vector of v w/ respect to B is the vector

$$[v]_B := (\alpha_1,\dots,\alpha_n) \in F^n.$$

□

Remark: Prop 2.2.2 asserts that the assignment $v \longmapsto [v]_B$ defines a function

$$\begin{aligned} V &\longrightarrow F^n \\ v &\longmapsto [v]_B \end{aligned}$$

Moreover, the function is a bijection w/ invers

$$\begin{aligned} F^n &\longrightarrow V \\ (\alpha_1,\dots,\alpha_n) &\longmapsto \sum_{i=1}^{n} \alpha_i b_i . \end{aligned}$$

We will prove that this is an isomorphism.

2.2.4 Proposition Let $B=(b_1,\dots,b_n)$ be an ordered basis for V. Then for all $u,v \in V$ and $\lambda \in F$

(a) $[u+v]_B = [u]_B + [v]_B$

(b) $[\lambda u]_B = \lambda [u]_B$.

Proof. Suppose $[u]_B = (\alpha_1,\dots,\alpha_n)$ and $[v]_B = (\beta_1,\dots,\beta_n)$. Then

$$\begin{aligned} u+v &= \sum_{i=1}^{n} \alpha_i b_i + \sum_{i=1}^{n} \beta_i b_i \\ &= \sum_{i=1}^{n} (\alpha_i+\beta_i) b_i \end{aligned}$$

This shows that

$$\begin{aligned} [u+v]_B &= (\alpha_1+\beta_1, \dots, \alpha_n+\beta_n) \\ &= (\alpha_1,\dots,\alpha_n) + (\beta_1,\dots,\beta_n) \\ &= [u]_B + [v]_B \end{aligned}$$

This proves (a). Part (b) is similar. $\square$

2.2.5 Corollary Let $\mathcal{B}$ be an ordered basis for V. Then the function

$$\begin{aligned} V &\longrightarrow F^n \\ v &\longmapsto [v]_{\mathcal{B}} \end{aligned}$$

is an isomorphism.

Proof. Follows immediately from Prop 2.2.4. $\square$

In particular, we now know that every finite-dimensional vector space V over F is isomorphic to $F^{\dim V}$.

2.3 Linear Maps and Bases

Linear maps are uniquely determined by their values on a basis.

2.3.1 Theorem Let $\mathcal{B} = (b_1, \dots, b_n)$ be an ordered basis for a vector space V and let W be any other vector space. For every sequence $c_1, \dots, c_n$ of vectors in W, there is a unique linear map

$$L: V \longrightarrow W$$

such that $L(b_i) = c_i$ for all $i = 1, \dots, n$.

Proof. Define $L: V \to W$ as follows. If $v \in V$ w/ $[v]_{\mathcal{B}} = (\alpha_1, \dots, \alpha_n)$, then $L(v) := \sum_{i=1}^{n} \alpha_i c_i$. This is a linear map: suppose $w \in V$ w/ $[w]_{\mathcal{B}} = (\beta_1, \dots, \beta_n)$.

Then $[v+u]_{\mathcal{B}} = (\beta_1+\alpha_1, \dots, \beta_n+\alpha_n)$ by Prop 2.2.4.

Then

$$\begin{aligned} L(v+u) &= \sum_{i=1}^{n} (\beta_i+\alpha_i) c_i \\ &= \sum_{i=1}^{n} (\beta_i c_i + \alpha_i c_i) \\ &= \sum_{i=1}^{n} \beta_i c_i + \sum_{i=1}^{n} \alpha_i c_i \\ &= L(v) + L(u). \end{aligned}$$

Similarly, $L(\lambda v) = \lambda L(v)$ for all $\lambda \in F$. So L is linear. Moreover, $L(b_i) = c_i$ since $[b_i]_{\mathcal{B}} = e_i$. This proves existence. Suppose $K : V \longrightarrow W$ is another linear map satisfying $K(b_i) = c_i$. Then

$$K(v) = K\left(\sum_{i=1}^{n} \alpha_i b_i\right) = \sum_{i=1}^{n} \alpha_i K(b_i) = \sum_{i=1}^{n} \alpha_i c_i = L(v).$$

This proves uniqueness.

$\square$

Remark: The choice of an ordered basis $\mathcal{B} = (b_1, \dots, b_n)$ and a list of vectors $c_1, \dots, c_n \in W$ amounts to the choice of a function

$$f : \mathcal{B} \longrightarrow W \quad (\text{namely } f(b_i) = c_i)$$

There is another function $\begin{aligned} c : \mathcal{B} &\xrightarrow{\subseteq} V \\ b &\longmapsto b \end{aligned}$. One can restate the theorem as follows: for any function $f : \mathcal{B} \longrightarrow W$, there is a unique linear map $L : V \longrightarrow W$ such that $L \circ c = f$, i.e. the diagram

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "\iota"'] \arrow[r, phantom, "\circlearrowleft"', shift right=15] & W \\ V \arrow[ur, dashed, "L"'] & \end{tikzcd} \qquad L\Big(\sum_{b\in B} \alpha_b b\Big) := \sum_{b\in B} \alpha_b L(b)$$

commutes.

Equivalently, the above statement asserts that there is a bijection of sets

$$\left\{\begin{matrix}\text{Linear maps}\\ L: V \to W\end{matrix}\right\} \xleftrightarrow{\sim} \left\{\begin{matrix}\text{functions}\\ f: B \to W\end{matrix}\right\} = W^B$$

$$\begin{aligned} L &\longmapsto L\circ\iota \\ L &\longleftarrow\!\shortmid\; f \end{aligned}$$

$\square$

2.3.2 Proposition Let V be a finite-dimensional vector space w/ basis $B = \{b_1, \dots, b_n\}$ and let $L: V \to W$ be a linear map. Then

(a) L is injective iff $L(B) = \{L(b_1), \dots, L(b_n)\}$ is independent in W.

(b) L is surjective iff $L(B)$ spans W.

(c) L is an isomorphism iff $L(B)$ is a basis for W.

Proof. Clearly, (c) follow from (a) and (b).

(a) For any $\alpha_1, \dots, \alpha_n \in F$ we have

$$\sum_{i=1}^{n} \alpha_i L(b_i) = L\Big(\sum_{i=1}^{n} \alpha_i b_i\Big).$$

If L is injective and LHS $=0$, then RHS $=0$ which implies $\sum_{i=1}^{n} \alpha_i b_i = 0$ since $\ker L = \{0\}$

But B is a basis, so $\alpha_1 = \cdots = \alpha_n = 0$. So $L(B)$ is independent. OTOH is $L(B)$ is indep. and RHS $=0$, then LHS $=0$ which implies $\alpha_1 = \cdots = \alpha_n = 0$. Hence $\sum_i \alpha_i b_i = 0$. So $\ker L = \{0\}$.

(b) ($\Rightarrow$) Let $w \in W$. Choose $v \in V$ such that $L(v) = w$. Write $v = \sum_i \alpha_i b_i$. Then

$$w = L(v) = \sum_i \alpha_i L(b_i) \in \mathrm{span}(L(B)).$$

($\Leftarrow$) Let $w \in W$. Since $L(B)$ spans, we can write

$$w = \sum_i \alpha_i L(b_i) = L\left(\sum_i \alpha_i b_i\right).$$

Set $v = \sum_i \alpha_i b_i$. Then $L(v) = w$. $\square$

Remark: one could also prove the following statement: if $S = \{s_1, \dots, s_n\}$ spans V and $L : V \longrightarrow W$, then $L(S)$ spans $\mathrm{im}(L)$. $\square$

2.3.3 Corollary Suppose V and W are finite-dimensional vector spaces w/ $\dim V = \dim W$. Let $L : V \longrightarrow W$ be a linear map. The following are equivalent:

(a) L is injective

(b) L is surjective

(c) L is an isomorphism.

Proof. Clearly, (a) and (b) $\Longleftrightarrow$ (c). Let B

be a basis for V. If L is injective, then $L(B)$ is independent by Prop 2.3.2 (a). But $L(B)$ has cardinality equal to $\dim W$ so $L(B)$ is a basis for W by Prop 1.4.15(a). So L is surjective by Prop 2.3.2(b). If L is surjective, then $L(B)$ spans W and has cardinality equal to $\dim W$. So $L(B)$ is a basis by 1.4.15(b). Hence, L is injective by Prop 2.3.2 (a). $\square$

2.3.4 Theorem Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.

Proof. Let V and W be f.d. vector spaces.

($\Rightarrow$) If $L: V \to W$ is an isomorphism and B is a basis for V, then $L(B)$ is a basis for W by Prop 2.3.2. Hence,

$$\dim V = |B| = |L(B)| = \dim W.$$

($\Leftarrow$) Suppose $\dim V = \dim W$. Then by Cor 2.2.5 $V \cong F^{\dim V}$ and $W \cong F^{\dim W}$. Hence $V \cong W$ since $F^{\dim V} = F^{\dim W}$. $\square$

Remark:

- Finite dimensional vector spaces are characterized up to isomorphism by their dimension.
- Each is isomorphic to F^n for some n.

So, why do we care about abstract f.d vector spaces at all? Why not just study F^n? The reason is that choosing an isomorphism $V \xrightarrow{\sim} F^n$ amounts to choosing an ordered basis for V. If $\mathcal{B} = (b_1, \dots, b_n)$ is a basis for V, then an iso is given by

$$\begin{array}{ccc} V & \xrightarrow{\sim} & F^n \\ v & \longmapsto & [v]_{\mathcal{B}} \\ b_i & \longleftarrow\!\shortmid & e_i \end{array}$$

One can think of this isomorphism as assigning a coordinate system to V. However, a different choice of basis results in a different coordinate system — and given an arbitrary abstract space V there is no natural or canonical basis. $\square$