1.3.8 Definition Let V be a vector space. Let Γ be any set and suppose that W_γ is a subspace of V for each $\gamma \in \Gamma$. Their **sum** is the subspace

$$\sum_{\gamma \in \Gamma} W_\gamma := \operatorname{Span}\Big(\bigcup_{\gamma \in \Gamma} W_\gamma\Big).$$

□

Note: The set Γ could be infinite. If $\Gamma = \{1, \dots, n\}$ is finite, then

$$\sum_{i=1}^{n} W_i = W_1 + \cdots + W_n = \{w_1 + \cdots + w_n \mid w_i \in W_i,\ i = 1, \dots, n\}.$$

This explains why its called the sum of subspaces!

1.3.9 Definition A subset S of a vector space V is called a **spanning set** for V if $\operatorname{Span}(S) = V$.

□

1.3.10 Example (spanning sets)

(a) $S = \{(1,0), (0,1)\}$ spans $\mathbb{R}^2$: $(x,y) = x(1,0) + y(0,1)$.

(b) $S = \{1, x, x^2, \dots\}$ spans $F[x]$ by definition.

(c) Consider $F^{\mathbb{N}} = \operatorname{Maps}(\mathbb{N}, F)$, the space of sequences in F. Define the sequences $(a_n^{(1)}), (a_n^{(2)}), \dots$ as follows

$$a_n^{(i)} := \begin{cases} 1, & n = i \\ 0, & n \neq i. \end{cases}$$

One might expect that $S = \{(a_n^{(i)}) \mid i \in \mathbb{N}\}$ spans $F^{\mathbb{N}}$. The problem is that taking span involves only finite sums. In fact, $\operatorname{span}(S)$ is the subspace of sequences w/ finitely many non zero entries.

□

Note: any superset of a spanning set is also a spanning set.

1.4 Bases and Dimension

1.4.1 Definition Let V be a vector space over F.

(a) The vectors $v_1, \dots, v_n \in V$ are **linearly independent** if $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

(b) A subset S of V is **linearly independent** if the vectors in any finite subset of S are linearly independent.

□

Note:
- An infinite set could be independent.
- An independent set does not contain 0.
- ϕ is independent

- v is independent iff $v \neq 0$
- v, w are independent iff v, w are not parallel.
- Any subset of an independent set is independent

1.4.2 Definition Let V be a vector space over F.

(a) The vectors $v_1, \dots, v_n \in V$ are **linearly dependent** if they are not independent.

(b) A subset S of V is **linearly dependent** if there exists a finite subset of S consisting of vectors that are dependent.

$\square$

Note: any superset of a dependent set is also dependent.

1.4.3 Proposition A subset S of a vector space V is dependent if and only if there exists $v \in S$ such that $v \in \operatorname{span}(S \setminus \{v\})$.

Proof. Exercise.

$\square$

1.4.4 Definition A **basis** for a vector space V is a linearly independent spanning set.

$\square$

The next proposition explains the importance of a basis.

1.4.5 Proposition Let $\mathcal{B}$ be a basis for a vector space V. Then every vector in V can be written uniquely as a linear combination of vectors in $\mathcal{B}$.

Proof. Let $v \in V$. Since $\mathcal{B}$ spans V, there is a finite index set I, a collection of vectors $\{b_i \mid i \in I\} \subseteq \mathcal{B}$, and a collection of scalars $\{\alpha_i \mid i \in I\} \subseteq F$ such

$$v = \sum_{i \in I} \alpha_i b_i .$$

Suppose there is another finite set J, another collection of vectors $\{b_j \mid j \in J\} \subseteq \mathcal{B}$ and scalars $\{\beta_j \mid j \in J\}$, such that

$$v = \sum_{j \in J} \beta_j b_j .$$

Then

$$0 = v - v = \sum_{i \in I \setminus J} \alpha_i b_i - \sum_{i \in J \setminus I} \beta_i b_i$$
$$+ \sum_{i \in I \cap J} (\alpha_i - \beta_i) b_i$$

The set $\{b_i \mid i \in I \cup J\} \subseteq B$ is independent. Hence, $\alpha_i = 0$ for all $i \in I \setminus J$, $\beta_i = 0$ for all $i \in J \setminus I$, and $\alpha_i = \beta_i$ for all $i \in I \cap J$.

□

1.4.6 Example (bases)

(a) F^n has the following basis, called the standard basis:

$$\begin{aligned} e_1 &:= (1,0,0,\dots,0) \\ e_2 &:= (0,1,0,\dots,0) \\ &\vdots \\ e_n &= (0,0,0,\dots,1). \end{aligned}$$

(b) Similarly, $F^{m \times n}$ has a basis E_{ij} where E_{ij} is the matrix with precisely one nonzero entry equal to 1 in the ith row, j^{th} column.

(c) $B = \{1, x, x^2, \dots\}$ is a basis for $F[X]$,

(d) $B = \{1, x, x^2, \dots, x^n\}$ is a basis $F_n[x]$.

(e) $B = \emptyset$ is a basis for $V = \{0\}$.

□

Note: $F[x]$ does not have a finite basis.

Question: Does every vector space have a basis? For example, does $F^{\mathbb{N}}$ have a basis?

1.4.7 Lemma If B is independent and $B \cup \{s\}$ is dependent, then $s \in \operatorname{span}(B)$.

Proof. If $B \cup \{s\}$ is dependent, then there is a nontrivial linear combo $\alpha_0 s + \alpha_1 b_1 + \cdots + \alpha_n b_n = 0$. Clearly, $\alpha_0 \neq 0$ because is independent. Hence,

$$s = -\frac{1}{\alpha_0} \sum_{i=1}^{n} \alpha_i b_i \in \operatorname{span}(B).$$

□

1.4.8 Proposition Let B be a subset of a vector space V. Then the following are equivalent:

- (a) B is a basis.
- (b) B is a maximal independent set.
- (c) B is a minimal spanning set.

Proof. ((a) ⇒ (b)) Suppose B is a basis. Then B is independent. Suppose C is another indep. set such that $B \subsetneq C$. Choose $v \in C \setminus B$. Then

$$v \in \operatorname{span}(B) \subseteq \operatorname{span}(C \setminus \{v\}).$$

By Prop 1.4.3, C is dependent — a contradiction.

((b) ⇒ (a)) Suppose B is maximal independent. Suppose $v \in V \setminus \operatorname{span}(B)$. Then $B \cup \{v\}$ is independent by Lemma 1.4.7. This contradicts maximality of B.

((b) ⇔ (c)) Exercise. □

1.4.9 Proposition (replacement) Let V be a vector space. If $v_1, \ldots, v_n \in V$ are independent and $S = \{s_1, \ldots, s_k\}$

is a finite spanning set, then for all $i \in \{1,\dots,n\}$ there exists $j \in \{1,\dots,k\}$ such that the vectors

$$v_1,\dots,v_{i-1}, s_j, v_{i+1},\dots,v_n$$

are also independent.

Proof. Suppose not. Then WLOG the statement is false for $i=1$. This means that for every $j \in \{1,\dots,k\}$, the vectors $s_j, v_2,\dots,v_n$ are dependent. Then $s_j \in \operatorname{span}\{v_2,\dots,v_n\}$ by Lemma 1.4.7. Thus,

$$v_1 \in \operatorname{Span}(S) \subseteq \operatorname{span}\{v_2,\dots,v_n\}.$$

This contradicts independence of $v_1,\dots,v_n$.

□

1.4.10 Proposition Let V be a vector space. If $v_1,\dots,v_n \in V$ are independent and $S=\{s_1,\dots,s_k\}$ is a spanning set, then $n \leq k$.

Proof. Using Prop 1.4.9, there exists $j_1 \in \{1,\dots,k\}$ such that $s_{j_1}, v_2,\dots,v_n$ are independent. By repeated application, there exists $j_2,\dots,j_n \in \{1,\dots,k\}$ such that $s_{j_1}, s_{j_2},\dots,s_{j_n}$ are independent. By independence, these vectors are all distinct. Thus,

$$n = |\{s_{j_1}, s_{j_2},\dots,s_{j_n}\}| \leq |S| = k.$$

□

1.4.11 Proposition Let V be a vector space. If V has a finite spanning set, then any independent set can be extended to a basis.

Proof. By Prop 1.4.10, any independent set is finite w/ cardinality $\leq |S|$. Let B be any independent set. If B is maximal independent, then B is a basis by Prop 1.4.8. If not, then there exists $v \in V \setminus B$ such that $B \cup \{v\}$ is independent. If $B \cup \{v\}$ is maximal indep, we are done. If not, repeat. This process ends after finitely many steps since $|S|$ is finite. $\square$

Note: Since $B = \emptyset$ is always independent, any vector space w/ a finite spanning set has a basis!

1.4.12 Proposition Let V be a vector space. If V has a finite spanning set, then any two bases for V have the same cardinality.

Proof. Suppose B and C are bases. Since B is spanning and C is independent, Prop 1.4.10 implies that $|C| \leq |B|$. Interchange the roles of B and C to get $|B| \leq |C|$. $\square$

1.4.13 Definition Let V be an F-vector space. The **dimension** $\dim_F V$ of V is defined to be the cardinality of a basis for V. $\square$

Note: • By Prop 1.4.12, a vector space w/ a finite spanning set has finite dimension and $\dim_F V$ is well-defined.

- Even if V has no finite spanning, it is possible to prove that any two base have the same cardinality.

1.4.14 Example (dimension)

(a) $\dim_F \{0\} = 0$

(b) $\dim_F F^n = n$

(c) $\dim_F F[x] = |\mathbb{N}| =: \aleph_0$

(d) $\dim_F F^{m \times n} = mn$

(e) Recall: we have subfields $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

• $\dim_{\mathbb{C}} \mathbb{C} = 1$ • $\dim_{\mathbb{R}} \mathbb{C} = 2$ (a basis $\{1, i\}$)

• $\dim_{\mathbb{R}} \mathbb{R} = 1$ • $\dim_{\mathbb{Q}} \mathbb{R} = \infty$

(Suppose B is a $\mathbb{Q}$-basis for $\mathbb{R}$. If $B = \{b_1, \dots, b_n\}$ is finite, then $\mathrm{Span}(B) = \{\alpha_1 b_1 + \cdots + \alpha_n b_n \mid \alpha_i \in \mathbb{Q}\}$ is a countable set. So $\mathrm{span}(B) \subsetneq \mathbb{R}$.)

1.4.15 Proposition Suppose V is a finite-dimensional vector space w/ $\dim V = n$. Then

(a) Any independent set of size n is a basis.

(b) Any spanning set of size n is a basis.

Proof. Combine Prop 1.4.8 and Prop 1.4.10. ☐

1.4.16 Proposition Let V be a finite-dimensional vector space and $W \subseteq V$ a subspace. Then W is finite-dimensional w/ $\dim W \leq \dim V$. Moreover, $W = V$ if and only if $\dim W = \dim V$.

Proof. Since V has a finite spanning set, so does W. Hence W is finite-dimensional. Let B be a basis for W. Then B is independent in V, hence can be extended to a basis for V by Prop 1.4.11. Thus, $\dim W \leq \dim V$. If $W = V$, then they share a basis so $\dim W = \dim V$. If $\dim W = \dim V$, then any basis B for W is a basis for V by Prop 1.4.15 (a). Thus, $W = \operatorname{Span}(B) = V$. ☐

Remark: Up to this point, everything we have proved relies only on the Zermelo - Fraenkel Set theory (ZF). This is a list of 8 axioms from which other theorems in set theory are proved. However, the theorem "every vector space has a basis" is independent from ZF in the sense that it cannot be proved from the 8 axioms alone. In order to prove such a theorem, we must assume a 9th axiom:

Axiom of Choice If $\{X_i\}_{i\in I}$ is a set of non-empty sets, then the Cartesian product

$$\prod_{i\in I} X_i := \left\{ f: I \to \bigcup_{i\in I} X_i \;\middle|\; f(i) \in X_i,\ \forall i \in I \right\}$$

is nonempty. $\square$

The elements of $\prod X_i$ are functions $f: I \to \bigcup X_i$ having the property that $f(i) \in X_i$ for all $i \in I$. So the Axiom of Choice asserts the existence of a "choice function". Given any collection of nonempty sets, we can form a new set by selecting one element from each. Seems reasonable, right?

So in Zermelo-Fraenkel set theory w/ choice (ZFC) one can prove the following theorem:

1.4.17 Theorem (Zorn's Lemma) Let $(P, \leq)$ be a non-empty partially ordered set. If every totally-ordered subset of P has an upper bound, then P has a maximal element. $\square$

Recall:

- A **partially ordered set** is a set P w/ a relation $\leq$ that is
  - reflexive $x \leq x \quad \forall x \in P$
  - transitive $x \leq y \wedge y \leq z \Rightarrow x \leq z$
  - antisymmetric $x \leq y \wedge y \leq x \Rightarrow x = y$
- **totally-ordered set** is a poset $(P, \leq)$ such that any 2 elements are comparable: $\forall x, y \in P$, $x \leq y$ or $y \leq x$.
- Let $Q \subseteq P$. An **upper bound** for Q is an element $x \in P$ s.t. $y \leq x \quad \forall y \in Q$.
- A **maximal element** of a poset P is an element $x \in P$ such that $x \leq y \Rightarrow x = y$.
  (E.g. a basis is a maximal element of the poset of all independent subsets of a v.s. V.)

In fact, Zorn's Lemma is actually equivalent to the Axiom of Choice! Let's use it!

1.4.18 Theorem **(existence of bases)** Let V be a vector space over F. Then V has a basis.

Proof. Here's the idea: by Prop 1.4.8 a basis for V is the same as a maximal element of the poset of all linearly independent subsets of V ordered by subset inclusion $\subseteq$. We can prove that this poset has a max'l element using Zorn's Lemma!

Let $L := \{ B \subseteq V \mid B \text{ is independent} \}$. Then $(L, \subseteq)$ is a partially ordered set. Moreover, L is nonempty since $\emptyset \in L$. Let $T \subseteq L$ be a totally ordered subset. We claim that $U := \bigcup_{B \in T} B$ is an upper bound for T. It suffices to show that $U \in L$, i.e., U is independent. Suppose

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$$

where $\alpha_i \in F$ and $v_i \in U$. Then there are $B_1, \ldots, B_n \in T$ such that $v_1 \in B_1, \ldots, v_n \in B_n$. Every finite subset of a totally ordered set has a maximum element, so there exists $j \in \{1, \ldots, n\}$ such that $B_i \subseteq B_j$ for all $i \in \{1, \ldots, n\}$. Thus, $v_1, \ldots, v_n \in B_j$. But B_j is independent, so $\alpha_1 = \cdots = \alpha_n = 0$. This proves that U is independent. Thus, by Zorn's lemma, L has a maximal element, i.e., a basis for V.

$\square$

Remark: One can actually prove that Thm 1.4.18 is actually equivalent to the Axiom of Choice. Most mathematicians work in ZFC because AOC seems reasonable. However, there are some bizarre consequences:

- Banach-Tarski paradox
- existence of non-measurable sets.