MATH 117: Daily Assignment 3

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Some hints for this assignment are written in the footnotes. See the daily assignment webpage for due dates, templates, and assignment description.

1. Compute $\operatorname{rank}(L)$ and $\operatorname{nullity}(L)$ for each of the following linear maps. You must justify your reasoning.

(a) $L : F_2[x] \to F_2[x]$, $L(a + bx + cx^2) = b + 2cx$ where F is any field.[1]

(b) $L : F^{2\times 2} \to F$, $L(A) = \operatorname{tr}(A)$ where F is any field. Here, $\operatorname{tr}(A)$ denotes the *trace* of the matrix A, i.e., the sum of the element on the main diagonal.

(c) $L : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}$, $L(a + b\sqrt{2}) = a + b$. Here, we view $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}$ as vector spaces over $\mathbb{Q}$.

(d) $L : F^3 \to F^3$, $L(a, b, c) = (0, b, c)$ where F is any field.

Solution. Your solution can go here. □

2. For each part, you are given a finite-dimensional F-vector space V with ordered basis B and a vector $v \in V$. Compute $[v]_B$.

(a) $V = \mathbb{R}^3$, $F = \mathbb{R}$, $B = ((1, 1, 2), (2, 3, 2), (1, 0, 1))$, $v = (-2, 1, 4)$.

(b) $V = F_2[x]$, F is any field with $\operatorname{ch} F \neq 2$[2], $B = (1 + x, 1 + x^2, x + x^2)$, $p(x) = a + bx + cx^2$ where $a, b, c \in F$.

(c) $V = \mathbb{C}$, $F = \mathbb{R}$, $B = (i, 1 - i)$, $v = x + iy$. Here i denotes the imaginary unit in $\mathbb{C}$.

(d) $V = \mathrm{Mag}_3(\mathbb{R})$ (see Daily 1.3), $F = \mathbb{R}$, $B = \left\{ \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right\}$, $v = \begin{pmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{pmatrix}$.

Solution. Your solution can go here. □

3. (a) Construct a linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ which maps the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ bijectively onto the plane $\{(x, y, z) \in \mathbb{R}^3 : 3x + 2y + z = 0\}$.[3]

(b) Construct a linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ which maps the plane $\{(x, y, z) \in \mathbb{R}^3 : x - y - z = 0\}$ bijectively onto the plane $\{(x, y, z) \in \mathbb{R}^3 : x - 3z = 0\}$.

Solution. Your solution can go here. □

[1] Hint: the answer depends on the characteristic of the field! Handle the case $\operatorname{ch} F = 2$ separately.
[2] This condition is required for the vectors in B to be independent.
[3] Theorem 2.3.1 my be useful here.