111B Section Week 3

Overview: Work on the following problems one at a time, either by yourself or in small-groups. After a sufficient amount of time has passed, we will discuss the solutions as a class. Attending section counts toward your participation grade.

1. Denote by $M_2(\mathbb{Z})$ the set of all 2×2 matrices with entries in $\mathbb{Z}$. It is a ring with the usual operations of matrix addition and matrix multiplication. Determine which of the following subsets of $M_2(\mathbb{Z})$ are subrings of $M_2(\mathbb{Z})$. Which subrings have a multiplicative identity?
   (a) $A = \{\left(\begin{smallmatrix} 0 & r \\ 0 & 0 \end{smallmatrix}\right) : r \in \mathbb{Z}\}$
   (b) $B = \{\left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right) : r \in \mathbb{Z}\}$
   (c) $C = \{\left(\begin{smallmatrix} s & r \\ 0 & t \end{smallmatrix}\right) : r, s, t \in \mathbb{Z}\}$
   (d) $D = \{\left(\begin{smallmatrix} 0 & 0 \\ r & 0 \end{smallmatrix}\right) : r \in \mathbb{Z}\}$
   (e) $E = \{\left(\begin{smallmatrix} r & r-s \\ 0 & s \end{smallmatrix}\right) : r, s \in \mathbb{Z}\}$
2. Let R be a ring and suppose $\{S_i : i \in I\}$ is a nonempty collection of subrings of R.
   (a) Prove that $\bigcap_{i \in I} S_i$ is a subring of R. In particular, this proves that the subring of R generated by a subset X of R is actually a subring.
   (b) Prove or disprove: $\bigcup_{i \in I} S_i$ is a subring of R.
3. Consider the ring $\mathbb{R}$ under the usual operations of addition and multipication. Define the set $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$. Using only the definitions, show that $\mathbb{Z}[\sqrt{2}]$ is the subring of $\mathbb{R}$ generated by $\mathbb{Z} \cup \{\sqrt{2}\}$.
4. Let R be a ring. An idempotent e in R is called *central* if $e \in Z(R)$. Show that the set $Re := \{re : r \in R\}$ is a subring of R. Does Re have a multiplicative identity?
5. Suppose that R is a division ring. Show that the center $Z(R)$ of R is a field.