111B Section Week 2

Overview: Work on the following problems one at a time, either by yourself or in small-groups. After a sufficient amount of time has passed, we will discuss the solutions as a class. Attending section counts toward your participation grade. In order to receive credit, you must write your name on this page and turn it in at the end of section.

1. Let A be any abelian group (written additively). Denote by $\mathrm{End}(A)$ the set of all group homomorphisms $f : A \to A$ (the *endomorphism ring* of A). Define addition $(+)$ and multiplication $(\cdot)$ on $\mathrm{End}(A)$ as follows: for all $f, g \in \mathrm{End}(A)$ and for all $a \in A$

$$(f+g)(a) := f(a) + g(a) \quad \text{and} \quad (f \cdot g)(a) := (f \circ g)(a).$$

   (a) Show that these operations make $\mathrm{End}(A)$ into a ring with identity. Describe the group of units $\mathrm{End}(A)^{\times}$.

   (b) Find a bijection $\Phi : \mathrm{End}(\mathbb{Z}) \to \mathbb{Z}$ having the following properties: for all $f, g \in \mathrm{End}(\mathbb{Z})$

      (i) $\Phi(\mathrm{Id}_A) = 1$,

      (ii) $\Phi(f+g) = \Phi(f) + \Phi(g)$, and

      (iii) $\Phi(f \cdot g) = \Phi(f)\Phi(g)$.

2. Let I be any non-empty set and let R_i be a ring for each $i \in I$.

   (a) Show that the Cartesian product of sets $\prod_{i \in I} R_i$ is a ring under componentwise addition and multiplication. We call $\prod_{i \in I} R_i$ the *direct product* of $\{R_i\}_{i \in I}$.

   (b) Find a necessary and sufficient condition for $\prod_{i \in I} R_i$ to have a 1.

3. Let $A := \prod_{n \in \mathbb{N}} \mathbb{Z}$, the direct product of copies of $\mathbb{Z}$ indexed by the positive integers. Consider the endomorphism ring $\mathrm{End}(A)$ of the underlying additive abelian group A.

   (a) Define $f : A \to A$ via $f((a_1, a_2, \ldots)) = (a_2, a_3, \ldots)$. Show that $f \in \mathrm{End}(A)$ and exhibit infinitely many right inverses for f.

   (b) Show that f is a left zero divisor, but not a right zero divisor.