Taylor Theorem (single-variable) Suppose $f:\mathbb{R}\to\mathbb{R}$ is of class C^K and $x_0\in\mathbb{R}$. Then

$$f(x_0+h) = \underbrace{f(x_0) + \sum_{i=1}^{K}\frac{1}{i!}f^{(i)}(x_0)h}_{} + R(x_0,h)$$

where $$\frac{R(x_0,h)}{h^K}\to 0 \quad \text{as} \quad h\to 0.$$ ∎

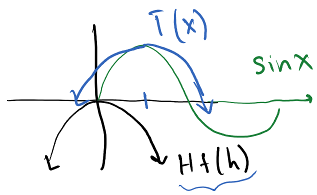
<u>Ex</u> $f(x)=\sin x$ on $[0,\pi]$. Locate all local extrema.

Find Critical Points: $f'(x)=\cos x = 0 \Rightarrow \boxed{x=\frac{\pi}{2}}$.

Compute Taylor Polynomial at $x_0=\frac{\pi}{2}$:

Set $x = x_0+h = \frac{\pi}{2}+h$ to get

$$f(x)\approx f(\tfrac{\pi}{2}) + f'(\tfrac{\pi}{2})(x-\tfrac{\pi}{2}) + \tfrac{1}{2}f''(\tfrac{\pi}{2})(x-\pi/2)^2$$
$$= \underbrace{1-\tfrac{1}{2}(x-\pi/2)^2}_{T} \qquad \text{Define } \underbrace{Hf(h) = -\tfrac{1}{2}h^2}_{\text{Hessian}}$$



$\pi/2$ is a max for $\sin x$ since the $\overbrace{\text{best quadratic approximation}}^{\text{according to the Taylor Thm,}}$ is concave down. We want to generalize this idea to C^k maps $f:\mathbb{R}^n\to\mathbb{R}$. ∎

Taylor Theorem (second-degree multivariable case) Suppose $f:\mathbb{R}^n\to\mathbb{R}$ of class C^2. Let $\vec{h}=(h_1,\dots,h_n)$ and let $\vec{x}_0\in\mathbb{R}^n$. Then

$$f(\vec{x}_0+\vec{h}) = f(\vec{x}_0) + \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\vec{x}_0)h_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f}{\partial x_i\partial x_j}(\vec{x}_0)h_ih_j + R_2(\vec{x}_0,\vec{h})$$

where $$\frac{\|R_2(\vec{x}_0,\vec{h})\|}{\|\vec{h}\|^2}\to 0 \quad \text{as} \quad \vec{h}\to\vec{0}.$$

<u>Proof</u> (sketch) Consider the line $c(t)=\vec{x}_0+t\vec{h}$. Compose w/ f:

$$g(t) = f(c(t)).$$

By the single-variable taylor theorem:

$$g(t) = g(0) + g'(0)t + \frac{1}{2}g''(0)t^2 \quad + R$$

$$\Rightarrow \quad f(\vec{x}_0 + \vec{h}) = g(1) = g(0) + g'(0) + \frac{1}{2}g''(0) \quad + R$$

compute using chain rule:

$$g'(t) = (f\circ c)'(t) = \nabla f(c(t)) \cdot c'(t)$$

$$c'(t) = h = (h_1, \dots, h_n)$$

$$= \nabla f(\vec{x}_0 + t\vec{h}) \cdot c'(t) \longleftarrow$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x}_0 + t\vec{h})\, h_i$$

$$g''(t) = \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial f}{\partial x_i}(\vec{x}_0 + t\vec{h})\right) h_i$$

$$= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_i \partial x_j}(\vec{x}_0 + t\vec{h})\, h_j\right) h_i$$

Evaluate at $t=0$ to get

$$f(\vec{x}_0 + \vec{h}) = g(1) = g(0) + g'(0) + \frac{1}{2}g''(0) \quad + R$$

$$= f(\vec{x}_0) + \sum_{i=1}^{n} \underline{\frac{\partial f}{\partial x_i}(x_0)}\, h_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \underline{\frac{\partial f}{\partial x_i \partial x_j}}(\vec{x}_0)\, h_i h_j \quad + R$$

We skip the statement about the remainder R.

<u>Ex</u> Taylor Polynomial for $f(x,y) = e^{x+y}$ at $x_0 = (0,0)$.

All partial derivatives at $(0,0)$ are equal to 1

$$T(x,y) = 1 + x + y + \frac{1}{2}(x^2 + xy + yx + y^2)$$

First-Derivative Test: Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable and suppose x_0 is a local max/min of f. Then x_0 is a critical point.

<u>Proof</u> Define $g(t) = \underline{(f\circ c)}(t)$ where $c(t) = \underline{x}_0 + th$ for any $h \in \mathbb{R}^n$.

Then $g(t)$ has a local max/min when $t=0$, so

$$0 = g'(0) = \nabla f(c(0)) \cdot c'(0)$$

$$= \nabla f(x_0) \cdot h \qquad\qquad h = (h_1, \dots, h_n)$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0) \cdot h_i$$

Since h is arbitrary, $\frac{\partial f}{\partial x_i}(x_0) = 0$ for all $i = 1, \dots, n$. So $Df(x_0) = 0$. $\blacksquare$

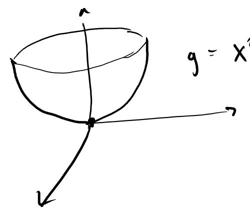
<u>Def</u> A quadratic function $g : \mathbb{R}^n \to \mathbb{R}$ of the form

$$g(h_1, \dots, h_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} h_i h_j, \qquad a_{ij} = a_{ji}$$

$$= [h_1, \dots, h_n] \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

(1) g is positive-definite if $g(h) \geq 0$ and $g(h) = 0 \iff h = 0$

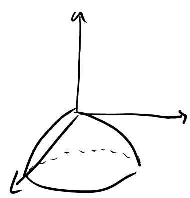
"concave up"



$g = x^2 + y^2$

(2) g is negative-definite if $g(h) \leq 0$ and $g(h) \iff h = 0$.

"concave down"

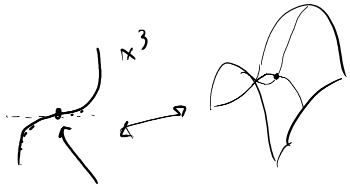


$g = -x^2 - y^2$

<u>Def</u> (Hessian) Let $f : \mathbb{R}^n \to \mathbb{R}$ be of class C^2 at $x_0 \in \mathbb{R}^n$. The Hessian is the quadratic fnc

$$Hf(x_0)(h_1, \dots, h_n) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)\, h_i h_j$$

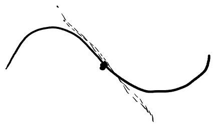
$$= \frac{1}{2}[h_1, \dots, h_n] \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \\ \frac{\partial^2 f}{\partial x_n \partial x_i} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$



$$\det \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial f}{\partial x \partial y} & \frac{\partial f}{\partial y y} \end{bmatrix} = 0$$

$\blacksquare$

Now assume x_0 is a critical point. By Taylor theorem, Now assume x_0 is a critical point. By Taylor theorem,



$$f(\vec{x}_0 + \vec{h}) = f(\vec{x}_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x}_0) h_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial f}{\partial x_i \partial x_j}(\vec{x}_0) h_i h_j + R$$

$$= f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot h + Hf(x_0)(h) + R$$

$$= f(\vec{x}_0) + Hf(x_0)(h) + R.$$

<u>Second-Der. Test</u> Let $f: \mathbb{R}^n \to \mathbb{R}$ be of class C^2 and let x_0 be a critical point.

(1) If $Hf(x_0)$ is posite-definite, then x_0 is a local min

(2) —— // —— negative-definite, —— // —— local max.