Chain Rule. Let \( f: \mathbb{R}^n \rightarrow \mathbb{R}^m \) and \( g: \mathbb{R}^m \rightarrow \mathbb{R}^p \) be differentiable at \( x_0 \in \mathbb{R}^n \) and \( y_0 = f(x_0) \), respectively. Then \( g \circ f \) is differentiable at \( x_0 \) and the Jacobian is

\[
D(g \circ f)(x_0) = Dg(y_0) \cdot Df(x_0)
\]

Special Cases

(i) (Derivative of a map along a path) Let \( f: \mathbb{R}^n \rightarrow \mathbb{R}^m \) and \( c: \mathbb{R} \rightarrow \mathbb{R}^m \) and form \( h: \mathbb{R} \rightarrow \mathbb{R}^m \), \( h = f \circ c \). By chain rule,

\[
\frac{dh}{dt} = Dh = Df \cdot Dc
\]

\[
= \left[ \frac{df_1}{dx_1}, \ldots, \frac{df_m}{dx_n} \right] \left[ \frac{dx_1}{dt} \ldots \frac{dx_n}{dt} \right]
\]

\[
= \sum_{i=1}^{n} \frac{dx_i}{dt} \frac{dx_i}{dt} = \nabla f \cdot c'(t).
\]

(ii) (Change of coordinates) Let \( f: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) and \( g: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) with component functions \( g(x,y) = (u(x,y), v(x,y)) \). The map \( g \) "changes the coordinates" from \((u,v)\) to \((x,y)\) (when we compose \( f \) with \( g \)).

(iii) (Polar coordinates) \( \Phi: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \)

\[
\Phi(r, \theta) = (r \cos \theta, r \sin \theta)
\]

(iv) (Spherical coordinates) \( \overrightarrow{\Psi}: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \)

\[
\Psi(r, \theta, \phi) = (r \sin \phi \sin \theta, r \sin \phi \cos \theta, r \cos \phi)
\]

Form \( h(x,y) = f(g(x,y)), \ g(x,y) = (u(x,y), v(x,y)) \). By chain rule,

\[
\begin{bmatrix}
\frac{dh}{dx} & \frac{dh}{dy}
\end{bmatrix} = Dh = Df \cdot Dg
\]

\[
= \begin{bmatrix}
\frac{df_1}{du} & \frac{df_1}{dv} \\
\frac{df_2}{du} & \frac{df_2}{dv}
\end{bmatrix}
\begin{bmatrix}
\frac{du}{dx} & \frac{du}{dy} \\
\frac{dv}{dx} & \frac{dv}{dy}
\end{bmatrix}
\]
\[
\begin{bmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y}
\end{bmatrix}
= U h = U + V y
\]
\[
= \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}
\end{bmatrix}
\]

Comparing entries gives formulas for \(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\) in terms of \(\frac{\partial l}{\partial x}, \frac{\partial l}{\partial y}\).

\[
\frac{\partial h}{\partial x} = \frac{\partial l}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial l}{\partial y} \quad \frac{\partial h}{\partial y} = \frac{\partial l}{\partial y} + \frac{\partial v}{\partial y} \frac{\partial l}{\partial y}
\]

Back to polar coordinates: \(f: \mathbb{R}^2 \rightarrow \mathbb{R}, \Phi(r,\theta) = (rcos\theta, rsin\theta)\). By Chain rule:

\[
\begin{bmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{bmatrix}
= \frac{\partial f \circ \Phi}{\partial \Phi} = \frac{\partial f}{\partial \Phi} \frac{\partial \Phi}{\partial \Phi} = \begin{bmatrix}
\frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\
\frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta}
\end{bmatrix}
\begin{bmatrix}
cos\theta & -r\sin\theta \\
sin\theta & r\cos\theta
\end{bmatrix}
\]

Comparing entries:

\[
\frac{\partial u}{\partial r} = \cos\theta \frac{\partial l}{\partial x} + \sin\theta \frac{\partial l}{\partial y} \quad \frac{\partial v}{\partial r} = -r\sin\theta \frac{\partial l}{\partial x} + r\cos\theta \frac{\partial l}{\partial y}
\]

Ex (Geometric interpretation of the Jacobian) Let \(f: \mathbb{R}^2 \rightarrow \mathbb{R}^2\) and \(c: \mathbb{R} \rightarrow \mathbb{R}^2\). Composing with \(f\) gives a new path \(p(t) = f \circ c(t)\). By chain rule,

\[
p'(t) = Dp(c(t)) = Df \circ c(t) \cdot Dc(t)
\]

\[
= Df(c(t)) \cdot C'(t) \quad (C'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix})
\]

This equation says the linear map \(L_{Df(c(t))}: \mathbb{R}^2 \rightarrow \mathbb{R}^2\), \(L([x]) = Df(c(t))[x]\), maps tangent vectors of \(c(t)\) to tangent vectors of \(p(t)\).

Ex Consider \(c(t) = (t,t)\) and \(\Phi: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \Phi(r,\theta) = (rcos\theta, rsin\theta)\). Compose to get a new path \(p(t) = \Phi \circ c(t) = (t\cos\theta, t\sin\theta)\) (spiral).
By chain rule, \[ p'(t) = \begin{bmatrix} \cos \theta - \sin \theta \\ \sin \theta + \cos \theta \end{bmatrix} \cdot \begin{bmatrix} 1 \end{bmatrix}. \] Then \[ p'(\pi) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix}. \]

**Gradient**

Let \( f: \mathbb{R}^n \to \mathbb{R} \) be differentiable. The gradient of \( f \) is the map \( \nabla f: \mathbb{R}^n \to \mathbb{R}^n \) defined by

\[ \nabla f = \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right). \]

The gradient assigns to each point \( x_0 \in \mathbb{R}^n \) a vector \( \nabla f(x_0) \in \mathbb{R}^n \).

**Directional Derivative**

Let \( f: \mathbb{R}^3 \to \mathbb{R} \) be differentiable and fix \( x_0, v \in \mathbb{R}^n \). Form the line \( c(t) = x_0 + tv \). Then \( c \) maps \( \mathbb{R} \to \mathbb{R}^3 \). The directional deriv. of \( f \) at \( x_0 \) in the direction of \( v \) is

\[ \left. \frac{d}{dt} f(c(t)) \right|_{t=0} = \frac{d}{dt} f(x_0 + tv) \bigg|_{t=0}. \]

By chain rule Special Case #1

\[ \left. \frac{d}{dt} f(c(t)) \right|_{t=0} = \nabla f(c(0)) \cdot c'(0) = \nabla f(x_0) \cdot v. \]

Usually \( v \) is taken to be a unit vector.

Then Assume \( \nabla f(x) \neq 0 \). Then \( \nabla f(x) \) points in the direction of steepest ascent of \( f \).

Proof Let \( n \) be a unit vector. The rate of change of \( f \) in the direction of \( n \) is given by

\[ \nabla f(x) \cdot n = ||\nabla f(x)|| \cdot ||n|| \cdot \cos \theta, \]

\[ = ||\nabla f(x)|| \cdot \cos \theta. \]

This is maximized when \( \theta = 0 \). If \( \theta = 0 \), \( \nabla f(x) \) and \( n \) are parallel.

**Tangent Plane to Level Curves**

Then let \( f: \mathbb{R}^3 \to \mathbb{R} \) be differentiable and consider the level surface \( S \) defined by \( f(x_1, x_2, x_3) = k \), \( k \in \mathbb{R} \). If \( (x_0, y_0, z_0) \in S \), then \( \nabla f(x_0, y_0, z_0) \)
Let \( c(t) \) be any path in \( S \) such that \( c(0) = (x_0, y_0, z_0) \). We need to show: \( \nabla f(x_0, y_0, z_0) \cdot \dot{c}(0) = 0 \).

We have

\[
\nabla f(x_0, y_0, z_0) \cdot \dot{c}(0) = \nabla f(c(0)) \cdot \dot{c}(0)
\]

\[
= \left. \frac{df}{dt} \right|_{t=0} = \left. \frac{df}{dt}(t_0) \right|_{t=0}
\]

\[
= \left. \frac{d}{dt} (k) \right|_{t=0} = 0.
\]

**Def** By the Theorem, the tangent plane to a level surface \( f(y_1, z) \) is given by the eq:

\[
\nabla f(x_0, y_0, z_0) \cdot (x-x_0, y-y_0, z-z_0) = 0
\]

This generalizes to maps \( f: \mathbb{R}^n \rightarrow \mathbb{R} \). In this case, \( f(x) = k \) defines an \( n \)-dimensional hypersurface. The same equation

\[
\nabla f(x) \cdot (x-x_0) = 0
\]

defines an \((n-1)\)-dimensional tangent space to the level set.

**Ex** \((n=2)\) Consider \( f(x,y) = x^2 + y^2 \) and form the level curves \( x^2 + y^2 = f(x,y) = k^2 \). These level curves are circles of radius \( k \).

The gradient is \( \nabla f(x,y) = (2x, 2y) \) so a vector normal to the circle at \((\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\) is \( \nabla f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = (\sqrt{2}, \sqrt{2}) \). The tangent space at this point is \((\sqrt{2}, \sqrt{2}) \cdot (x-\frac{\sqrt{2}}{2}, y-\frac{\sqrt{2}}{2}) = 0 \)

\[
\Rightarrow \sqrt{2}x + \sqrt{2}y = \sqrt{2} \Rightarrow y = \frac{\sqrt{2} - x}{\sqrt{2}}
\]