Saturday, November 14, 2020 11:36 AM

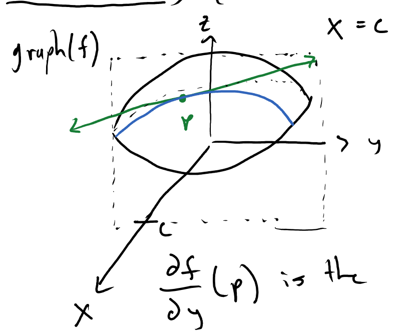
## Derivative of a map $f: \mathbb{R}^n \to \mathbb{R}^n$

**Partial Derivatives**: Let $f: \mathbb{R}^n \to \mathbb{R}$. The partial derivative of $f$ w.r.t. $x_i$ is the following limit (if it exists)

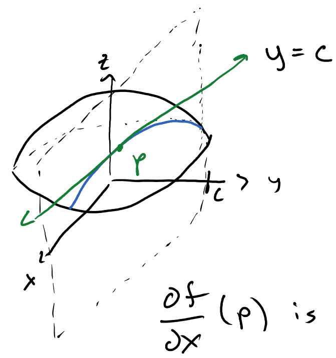
$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \cdots, x_i + h, \cdots, x_n) - f(x_1, \cdots, x_n)}{h}$$

If $f: \mathbb{R}^n \to \mathbb{R}^m$, then we can write $f = (f_1, \dots, f_m)$, $f_i: \mathbb{R}^n \to \mathbb{R}$. So we have partial derivatives $\frac{\partial f_i}{\partial x_j}$, $i = 1, \dots, m$, $j = 1, \dots, n$. ■

**Geometrically** ($n=2$, $m=1$, $f: \mathbb{R}^2 \to \mathbb{R}$)




$\frac{\partial f}{\partial y}(p)$ is the slope of this tangent line

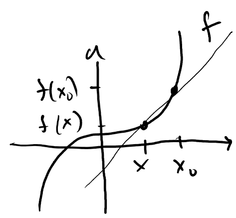



$\frac{\partial f}{\partial x}(p)$ is the slope of this tangent line. ■

**Prototype #1** ($n = m = 1$, $f: \mathbb{R} \to \mathbb{R}$)

The derivative of $f$ is the limit (if it exists)

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$




$m = \frac{f(x) - f(x_0)}{x - x_0}$ (slope of secant line)

Equivalently,

$$0 = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} - \lim_{x \to x_0} f'(x_0)$$

$$= \lim_{x \to x_0} \left( \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right)$$

$$= \lim_{x \to x_0} \frac{f(x) - \underbrace{(f(x_0) + f'(x_0)(x - x_0))}_{\text{is a line}}}{x - x_0}$$

This just means $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ when $x$ is near $x_0$. ■

**Prototype #2** ($n = 2$ $m = 1$, $f: \mathbb{R}^2 \to \mathbb{R}$)

We expect that the best linear approximation to $f$ at $(x_0, y_0) \in \mathbb{R}^2$ will be a tangent plane. Any nonvertical plane containing $(x_0, y_0, z_0)$ can be written

$$P: \quad z - z_0 = a(x-x_0) + b(y-y_0)$$

Assume $P$ is tangent to $f$ at $(x_0, y_0, f(x_0,y_0))$.

Fix $x = K$, then $z = b(y-y_0) + \underbrace{(a(K-x_0) + z_0)}_{\text{constant}}$ is a line in the x-direction with slope $b$. So $b = f_y(x_0,y_0)$

Similarly, $a = f_x(x_0,y_0)$. Also, $z_0 = f(x_0,y_0)$ if $P$ contains $(x_0, y_0, f(x_0,y_0))$.

So the tangent plane <u>should be</u>

$$z - f(x_0,y_0) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

So, we make the following definition:

Let $f: \mathbb{R}^2 \to \mathbb{R}$. We say that $f$ is <u>differentiable</u> at $(x_0,y_0) \in \mathbb{R}^2$ if $\frac{\partial f}{\partial x}(x_0,y_0)$ and $\frac{\partial f}{\partial y}(x_0,y_0)$ exist <u>and</u> the following limit is zero:

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - (f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0))}{\|(x,y) - (x_0,y_0)\|} = 0.$$

The numerator can be rewritten:

$$\underbrace{f(x,y) - (f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0))} = f(x,y) - \left(f(x_0,y_0) + \overbrace{\begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \end{bmatrix}}^{Df(x_0,y_0)} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}\right)$$

The <u>derivative</u> of $f$ at $(x_0,y_0)$ is the $1\times 2$ matrix $Df(x_0,y_0)$. ∎

<u>General Case</u> Let $f: \mathbb{R}^n \to \mathbb{R}^m$. We say that $f$ is <u>differentiable</u> at $x_0 \in \mathbb{R}^n$ if all the partial derivatives $\frac{\partial f_i}{\partial x_j}$, $\begin{matrix} i=1,\dots,n \\ j=1,\dots,m \end{matrix}$ exist and

$$\lim_{x\to x_0} \frac{\|\overbrace{f(x)}^{m\times 1} - (\overbrace{f(x_0)}^{m\times 1} + \overbrace{Df(x_0)}^{m\times n}\overbrace{(x-x_0)}^{n\times 1})\|}{\underbrace{\|x-x_0\|}_{n\times 1}} = 0.$$

where

$$Df(x_0) = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \cdots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0) & \cdots & \frac{\partial f_m}{\partial x_n}(x_0) \end{bmatrix}}_{n} \Big\} m.$$

$Df(x_0)$ is called the <u>derivative</u>, <u>differential</u>, or <u>Jacobian</u>. ∎

<u>The Jacobian as a linear map</u>: Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be differentiable. Let $D = Df(x_0)$.

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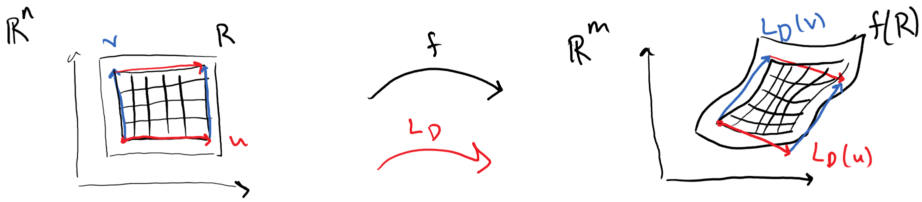
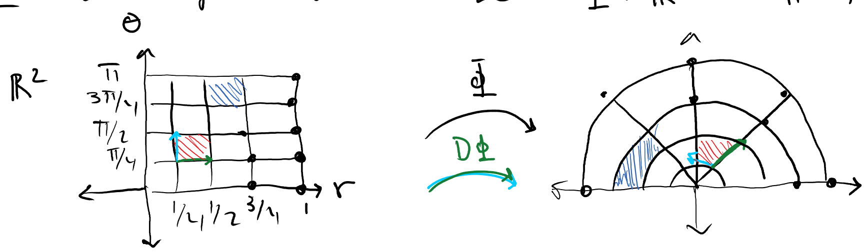
D is a matrix of real numbers. We can view D as a linear map:

$$L_D : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$y \longmapsto Dy$$




Ex Polar change of Coordinates: Let $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$.




The derivative of $\Phi$: 

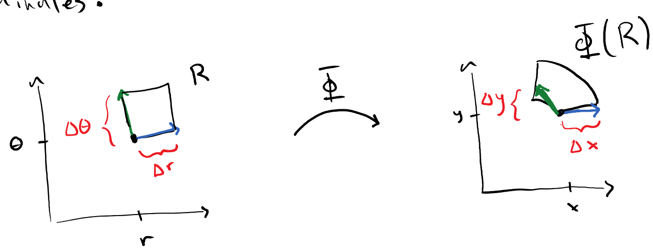
$$D\Phi(r,\theta) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

A fact from linear algebra: Let A be a $2\times 2$ matrix. If $x, y \in \mathbb{R}^2$, then

(area of parallelogram spanned by $Ax, Ay$) = (area of parallelogram spanned by $x, y$) $|\det A|$.

In our example, $\det(D\Phi(r,\theta)) = r\cos^2\theta + r\sin^2\theta$
$= r$.

This is the ratio of the areas of the approximating parallelograms under the polar change of coordinates.




So, $$r = \frac{\text{Area}(\Phi(R))}{\text{Area}(R)} = \frac{\Delta x \Delta y}{\Delta\theta \Delta r}.$$

Application $\displaystyle\iint_D f(x,y)\,dx\,dy = \iint_D f(r,\theta)\,|\det \Phi|\,dr\,d\theta$

$$= \iint_D f(r,\theta)\cdot r\,dr\,d\theta$$