Paths and Curves

**Def.** A path is a map \( c: I \rightarrow \mathbb{R}^n \) where \( I \subseteq \mathbb{R} \) is an interval of real numbers.

**Ex.**
1. **Circle of radius** \( R \), \( c: [0,2\pi] \rightarrow \mathbb{R}^2 \)
   
   \[ c(t) = (R \cos t, R \sin t) \]

2. **Cylindrical Helix:** a helix lying on the cylinder \( x^2 + y^2 = R^2 \)
   
   \[ H: [0,4\pi] \rightarrow \mathbb{R}^3 \]
   
   \[ c(t) = (R \cos t, R \sin t, t) \]

3. **Conic Helix:** a helix lying on the surface of \( x^2 + y^2 = z^2 \)
   
   \[ c(t) = (r(t) \cos t, r(t) \sin t, t) \]

   If \( c \) lies on \( x^2 + y^2 = z^2 \), then \( r^2(t) = t^2 \rightarrow r(t) = \pm t \)

4. **Intersection of two surfaces:** \( x^2 + y^2 = 4 \) and \( z = \sin(5x) \)
   
   If \( c(t) = (x(t), y(t), z(t)) \) lies in the cylinder,
then \( x(t) = 2 \cos t \) \( y = 2 \sin t \). Then
\[
\begin{align*}
\dot{z}(t) &= \sin (5x(t)) \\
&= \sin (5 \cdot 2 \cos t).
\end{align*}
\]
**Lemma** Suppose \( r : I \rightarrow \mathbb{R}^n \) satisfies \( \| r(t) \| = k \) for all \( t \in I \). Then \( r(t), r'(t) = 0 \) for all \( t \in I \).

**Proof**
\[
0 = \frac{d}{dt} k^2 = \frac{d}{dt} \| r(t) \|^2 = \frac{d}{dt} r(t) \cdot r(t) = r(t) r'(t) + r'(t) \cdot r(t) = 2 r(t) \cdot r'(t)
\]

**Def** Let \( r : I \rightarrow \mathbb{R}^3 \) be a path. For each \( t \in I \), define

1. **(Unit Tangent Vector)**
   \[
   T(t) = \frac{r'(t)}{\| r'(t) \|}
   \]
   This is a unit vector tangent to \( r(t) \).

2. **(Unit Normal Vector)**
   \[
   N(t) = \frac{T'(t)}{\| T'(t) \|}
   \]
   Note that \( \| T(t) \| = 1 \) for all \( t \in I \). By the Lemma, \( N(t) \) unit vector that is perpendicular to \( T(t) \).

3. **(Unit Binormal Vector)**
   \[
   B(t) = T(t) \times N(t)
   \]
   \( B(t) \) is a unit vector to both \( T(t) \) and \( N(t) \).
At each point on the curve, the vectors $T, N, B$ form a non-inertial frame of reference.
**Definition:** Let $r: I \to \mathbb{R}^3$ be a curve and $p = r(t_0)$ a point on the curve.

**Summary:** $r: I \to \mathbb{R}^3$ a curve

$$
T(t) = \frac{r'(t)}{\|r'(t)\|}, \quad N(t) = \frac{T'(t)}{\|T'(t)\|}, \quad B(t) = T(t) \times N(t)
$$

**Normal Plane:** the plane containing $p$ and perpendicular to $T(t_0)$

**Osculating Plane:** the plane containing $p$ and perpendicular to $B(t_0)$

**Example:** Osculating plane for the helix $h(t) = (\cos t, \sin t, t)$ when $t = \pi/2$.

**Solution:** A point in the plane is $h(\pi/2) = (0, 1, \pi/2)$.

$$
T(t) = \frac{h''(t)}{\|h''(t)\|} = \frac{(-\sin t, \cos t, 1)}{\sqrt{\sin^2 t + \cos^2 t + 1}} = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1)
$$

$$
N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{\frac{1}{\sqrt{2}}(-\cos t, -\sin t, 0)}{\sqrt{\cos^2 t + \sin^2 t + 0^2}} = (-\cos t, -\sin t, 0)
$$

$$
\sqrt{2} B(\pi/2) = \sqrt{2} T(\pi/2) \times N(\pi/2) = \begin{vmatrix}
 1 & i & k \\
-1 & 0 & i \\
0 & -1 & 0 \\
\end{vmatrix} = (1, 0, 1)
$$

This is our normal vector for the osculating plane.
\[(1, 0, 1) \cdot ((x, y, z) - (0, 1, \pi/2)) = 0\]

\[\Rightarrow \quad x + z = \frac{\pi}{2}\]

The osculating plane “measures” how close a small section of a curve is to being planar. In our picture, points near \((0, 1, \pi/2)\) are very close to the plane \(x + z = \pi/2\).
**Chapter 12**

**Curvature**

**Def** Let \( r: I \rightarrow \mathbb{R}^3 \) be a path. The curvature of \( r(t) \) is the quantity

\[
K(t) = \frac{||T'(t)||}{||r'(t)||^3} = \frac{||r'(t) \times r''(t)||}{||r'(t)||^3}
\]

The curvature \( K \) describes how far away a curve is from being straight (locally).

**Ex** The helix has constant curvature \( (h(t) = (2\cos t, 2\sin t, t)) \)

Solution

\[
h'(t) = (-2\sin t, 2\cos t, 1)
\]

\[
||h'(t)|| = \left(16\sin^2 t + 4\cos^2 t + 1\right)^{1/2} = \sqrt{17}
\]

\[
h''(t) = (-2\cos t, -2\sin t, 0)
\]

\[
h'(t) \times h''(t) = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
-2\sin t & 2\cos t & 1 \\
-2\cos t & -2\sin t & 0 \\
\end{vmatrix}
\]

\[
= K\left(16\sin^2 t + 4\cos^2 t + 1\right) - \left((-2\sin t)\hat{j} - (2\cos t)\hat{i}\right)
\]

\[
= (2\sin t, 2\cos t, 4)
\]

\[
||h'(t) \times h''(t)|| = \sqrt{4\sin^2 t \cdot 4\cos^2 t + 4} = \sqrt{16\sin^2 t \cdot \cos^2 t + 4} = 2\sqrt{5}
\]

So

\[
K(t) = \frac{2\sqrt{5}}{17^{1/2}} = \frac{2}{\sqrt{5}}
\]

**Ex** Curvature of the parabola \( p(t) = (t, t^2, 0) \) at \( t = 0 \).

\[
p'(t) = (1, 2t, 0) \quad \Rightarrow \quad ||p'(0)|| = 1
\]

\[
p''(t) = (0, 2, 0)
\]

\[
p'(0) \times p''(0) = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 2 & 0 \\
0 & 2 & 0 \\
\end{vmatrix} = \begin{vmatrix}
2 & 0 \\
0 & 2 \\
\end{vmatrix} = 4
\]

\[
||p'(0) \times p''(0)|| = 2 \quad \Rightarrow \quad K(0) = \frac{2}{1} = 2
\]
Fact: The curvature defines the "best approximating circle" at a point on a curve. This is the circle that has radius \( r = \frac{1}{\kappa} \), lies tangent to the curve, and is contained in the osculating plane.

\[
r = \frac{1}{\kappa(t_0)} = \frac{1}{2}
\]