Paths and Curves

Chapter 12

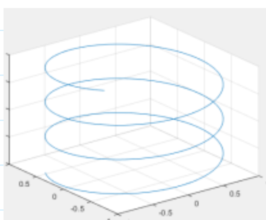
Def A path is a map $c: I \to \mathbb{R}^n$ where $I \subseteq \mathbb{R}$ is an interval of real numbers.

Ex (1) Circle of radius R, $c: [0, 2\pi] \to \mathbb{R}^2$

$$c(t) = (R\cos t, R\sin t)$$

(2) Cylindrical Helix: a helix lying on the cylinder $\underline{x^2+y^2=R^2}$

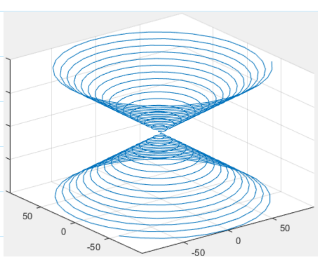
$$H: [0, 4\pi] \to \mathbb{R}^3, \quad c(t) = (R\cos t, R\sin t, t)$$



(3) Conic Helix: a helix lying on the surface of $x^2+y^2=z^2$

$$c(t) = (r(t)\cos t, r(t)\sin t, t)$$

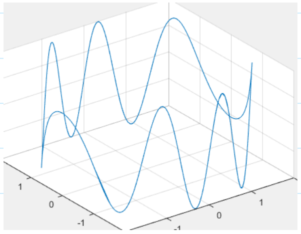
If c lies on $x^2+y^2=z^2$, then $r^2(t) = t^2 \Rightarrow r(t) = \pm t$



(4) Intersection of two surfaces: $\underline{x^2+y^2=4}$ and $z = \sin(5x)$

If $c(t) = (x(t), y(t), z(t))$ lies in the cylinder, then $\quad x(t) = 2\cos t \quad y = 2\sin t.$ Then

$$z(t) = \sin(5x(t))$$
$$= \sin(5 \cdot 2\cos t).$$

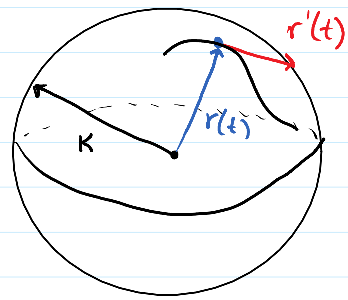


Geometry of Space curves (paths in $\mathbb{R}^3$)

Lemma Suppose $r : I \to \mathbb{R}^n$ satisfies $\|r(t)\| = K$ for all $t \in \mathbb{R}$.
Then $r(t) \cdot r'(t) = 0$ for all $t \in \mathbb{R}$.

Proof

$$
\begin{aligned}
0 = \frac{d}{dt} K^2 &= \frac{d}{dt} \|r(t)\|^2 \\
&= \frac{d}{dt} r(t) \cdot r(t) \\
&= r(t) \cdot r'(t) + r'(t) \cdot r(t) \\
&= 2\, r(t) \cdot r'(t) \quad \blacksquare
\end{aligned}
$$



Def Let $r : I \longrightarrow \mathbb{R}^3$ be a path. For each $t \in I$, define

(1) (Unit Tangent Vector)

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

This is a unit vector tangent to $r(t)$.

(2) (Unit Normal Vector)

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

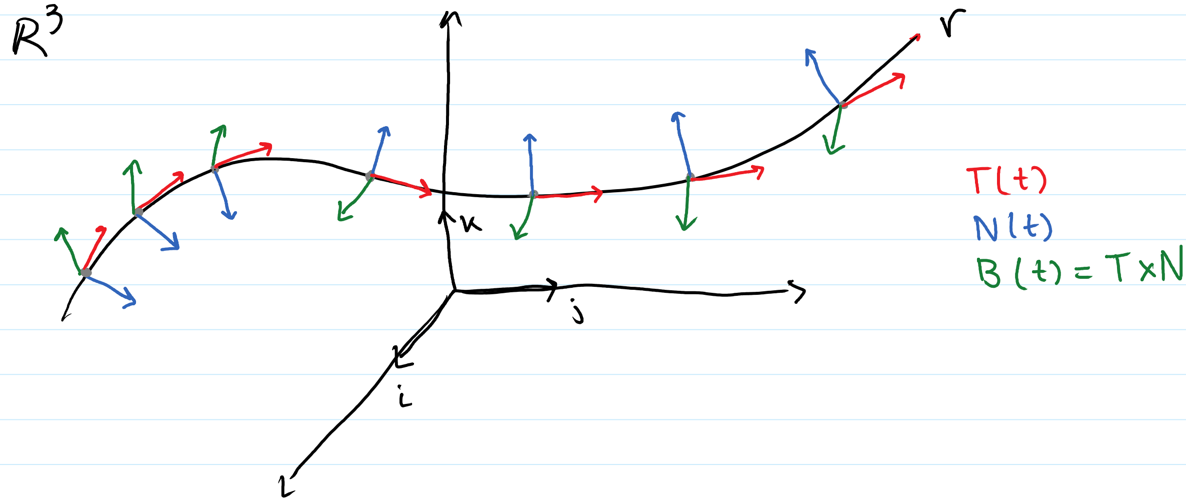
(*) $\|T'(t)\| \neq 0$ as long as the curvature is non zero.

Note that $\|T(t)\| = 1$ for all $t \in I$. By the Lemma $N(t)$ unit vector that is perpendicular to $T(t)$.

(3) (Unit Binormal Vector)

$$B(t) = T(t) \times N(t)$$

$B(t)$ is a unit vector to both $T(t)$ and $N(t)$.



At each point on the curve, the vectors T, N, B form a non-inertial frame of reference.

Normal Plane / Osculating Plane

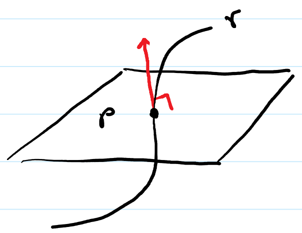
Def Let $r: I \to \mathbb{R}^3$ be a curve and $p = r(t_0)$ a point on the curve.

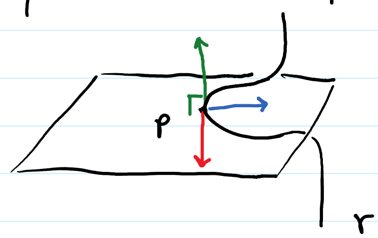
Summary: $r: I \to \mathbb{R}^3$ a curve

$T(t) = \frac{r'(t)}{\|r'(t)\|}$ $\quad N(t) = \frac{T'(t)}{\|T'(t)\|}$ $\quad B(t) = T(t) \times N(t)$

Normal Plane: the plane containing p and perpindicular to $T(t_0)$

Osculating Plane: the plane containing p and perpindicular to $B(t_0)$





EX: Osculating plane for the helix $h(t) = (\cos t, \sin t, t)$ when $t = \pi/2$.

Solution A point in the plane is $h(\pi/2) = (0, 1, \pi/2)$.

$$T(t) = \frac{h'(t)}{\|h'(t)\|} = \frac{(-\sin t, \cos t, 1)}{\sqrt{\sin^2 t + \cos^2 t + 1}} = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1)$$

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{\cancel{\frac{1}{\sqrt{2}}}(-\cos t, -\sin t, 0)}{\cancel{\frac{1}{\sqrt{2}}}\sqrt{\cos^2 t + \sin^2 t + 0^2}} = (-\cos t, -\sin t, 0)$$

$$\sqrt{2}B(\pi/2) = \sqrt{2}T(\pi/2) \times N(\pi/2) = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}$$

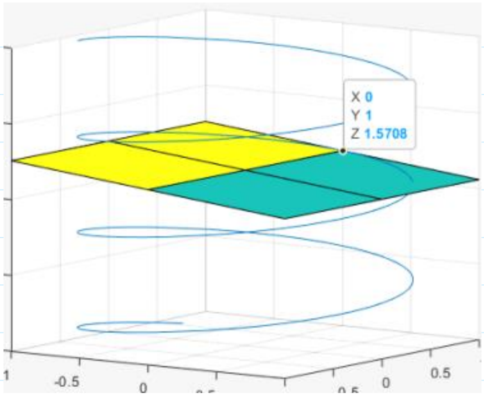
$$= (1, 0, 1)$$

This is our normal vector for the osculating plane.

$$(1,0,1)\cdot\left((x,y,z)-(0,1,\pi/2)\right)=0$$

$\Rightarrow$

$\Rightarrow$ $\boxed{x+z=\pi/2}$



The osculating plane "measures" how close a small section of a curve is to being planar. In our picture, points near $(0,1,\pi/2)$ are very close to the plane $x+z=\pi/2$.

$\blacksquare$

Def Let $r : I \to \mathbb{R}^3$ be a path. The curvature of $r(t)$ is the quantity

needs proof

$$\underline{K}(t) = \frac{\|T'(t)\|}{\|r'(t)\|} \overset{(*)}{=} \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

The curvature K describes how far away a curve is from being straight (locally)

Ex The helix has constant curvature $(h(t) = (2\cos t, 2\sin t, t))$

Solution

$$h'(t) = (-2\sin t, 2\cos t, 1) \qquad \|h'(t)\|^3 = \left(\sqrt{4\sin^2 t + 4\cos^2 t + 1}\right)^3$$

$$h''(t) = (-2\cos t, -2\sin t, 0) \qquad = \sqrt{5}^{\,3} = 5^{3/2}$$

$$h'(t) \times h''(t) = \begin{vmatrix} i & j & k \\ -2\sin t & 2\cos t & 1 \\ -2\cos t & -2\sin t & 0 \end{vmatrix} \begin{matrix} + \\ - \\ + \end{matrix}$$

$$= K(\underbrace{4\sin^2 t + 4\cos^2 t}) - 1\left(i(-2\sin t) - j(-2\cos t)\right)$$

$$= (2\sin t, -2\cos t, 4)$$

$$\|h'(t) \times h''(t)\| = \sqrt{4\sin^2 t \; 4\cos^2 t + 16} = \sqrt{20} = 2\sqrt{5}$$

$$\text{So} \quad K(t) = \frac{2\sqrt{5}}{5^{3/2}} = \boxed{\frac{2}{5}}$$

★ The curvature of a helix $(a\cos t, a\sin t, bt)$ is $K = \frac{a}{a^2+b^2}$.

Ex Curvature of the parabola $p(t) = (t, t^2, 0)$ at $\underline{t=0}$.

$$p'(t) = (1, 2t, 0) \implies \|p'(0)\|^3 = \|(1,0,0)\| = 1$$

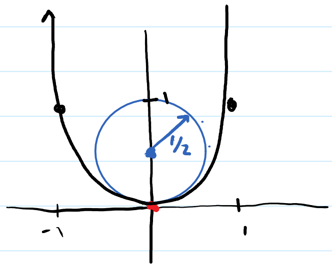
$$p''(t) = (0, 2, 0)$$

$$\hat{i} \times 2j = 2(i \times j) = 2K$$

$$\implies p'(0) \times p''(0) = (1,0,0) \times (0,2,0) = (0,0,2)$$

$$\implies \|p'(0) \times p''(0)\| = 2 \implies K(0) = \frac{2}{1} = 1$$

Fact The curvature defines the "best approximating circle" at a point on a curve. This is the circle that has radius $r = 1/K$, lies tangent to the curve, and is contained in the osculating plane.



$$r = \frac{1}{K(0)} = \frac{1}{2}$$