Determinants of Large Matrices

How can we compute determinates like

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

efficiently?

(1) Recursively using cofactor expansion:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

<u>Ex</u> Triangular Matrices

$$\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} \overset{(1)}{=} a_{11}\begin{vmatrix} a_{22} & \cdots & a_{2n} \\ 0 & \ddots & \vdots \\ 0 \cdots 0 & & a_{nn} \end{vmatrix}$$

$$\overset{(1)}{=} a_{11}a_{22}\begin{vmatrix} a_{33} & \cdots & a_{3n} \\ 0 & \ddots & \vdots \\ 0 \cdots 0 & & a_{nn} \end{vmatrix}$$

$n-3$ more times $\Big\{\quad \vdots$

$$= a_{11}a_{22}\cdots a_{nn} \qquad \blacksquare$$

<u>Ex</u> Compute the determinant:

$$\det\begin{bmatrix} {}^{+}x & y & 0 & 0 & 0 \\ {}^{-}0 & x & y & 0 & 0 \\ {}^{+}0 & 0 & x & y & 0 \\ {}^{-}0 & 0 & 0 & x & y \\ {}^{+}y & 0 & 0 & 0 & x \end{bmatrix} = x\underbrace{\begin{vmatrix} x & y & 0 & 0 \\ 0 & x & y & 0 \\ 0 & 0 & x & y \\ 0 & 0 & 0 & x \end{vmatrix}}_{\text{upper triangular}} + y\underbrace{\begin{vmatrix} y & 0 & 0 & 0 \\ x & y & 0 & 0 \\ 0 & x & y & 0 \\ 0 & 0 & x & y \end{vmatrix}}_{\text{lower triangular}}$$

$$= x\cdot x^4 + y\,y^4 = x^5 + y^5. \qquad \blacksquare$$

(2) Row/Column Operation: Let A be an $n \times n$ matrix

(a) Switch two row of A (or columns) $[R_i \longleftrightarrow R_j]$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

(b) Multiply a row (or column) by a nonzero constant

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_1 \longleftarrow 2R_1} \begin{bmatrix} 2 & 4 \\ 3 & 4 \end{bmatrix} \qquad [R_i \longleftarrow cR_i]$$

(c) Add a multiple of one row (or column) to another

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{C_2 \longleftarrow C_2 + 6C_1} \begin{bmatrix} 1 & 2+6\cdot 1 \\ 3 & 4+6\cdot 3 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 3 & 22 \end{bmatrix} \qquad [R_i \longleftarrow R_i + cR_j]$$

Row/Column Operations + Determinants

(a) Suppose B is obtained from A by operation (a).

Then $\det(B) = -\det(A)$.

(b) Suppose B is obtained from A by operation (b).

Then $\det(B) = c\det(A)$.

(c) Suppose B is obtained from A by operation (c).

Then $\det(B) = \det(A)$.

Ex

$$\det \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 4 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ -1 & 0 & 2 & -3 \end{bmatrix} \overset{R_1 \longleftarrow R_1 + R_4}{\overset{(c)}{=}} \det \begin{bmatrix} \overset{+}{0} & \overset{-}{0} & \overset{+}{4} & 0 \\ 1 & 4 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ -1 & 0 & 2 & -3 \end{bmatrix}$$

$$\text{cofactor} = 4\begin{vmatrix} 1 & 4 & 1 \\ 1 & 2 & 1 \\ -1 & 0 & -3 \end{vmatrix}$$

$R_1 \leftarrow R_1 - R_2$

$$\overset{(c)}{=} 4\begin{vmatrix} {}^{+}0 & {}^{-}2 & {}^{+}0 \\ 1 & 2 & 1 \\ -1 & 0 & -3 \end{vmatrix}$$

$$= 4(-2)\begin{vmatrix} 1 & 1 \\ -1 & -3 \end{vmatrix}$$

$$= -8(-3-(-1)) = 16 \qquad \blacksquare$$

Ex

$$\det\begin{bmatrix} x & y & y & y & y \\ y & x & y & y & y \\ y & y & x & y & y \\ y & y & y & x & y \\ y & y & y & y & x \end{bmatrix} = \begin{vmatrix} x & y & y & y & y \\ y-x & x-y & 0 & 0 & 0 \\ 0 & y-x & x-y & 0 & 0 \\ 0 & 0 & y-x & x-y & 0 \\ 0 & 0 & 0 & y-x & x-y \end{vmatrix}$$

Steps

1. $R_5 \leftarrow R_5 - R_4$
2. $R_4 \leftarrow R_4 - R_3$
3. $R_3 \leftarrow R_3 - R_2$
4. $R_2 \leftarrow R_2 - R_1$

$$= \begin{vmatrix} x+4y & 4y & 3y & 2y & y \\ 0 & x-y & 0 & 0 & 0 \\ 0 & 0 & x-y & 0 & 0 \\ 0 & 0 & 0 & x-y & 0 \\ 0 & 0 & 0 & 0 & x-y \end{vmatrix}$$

Steps

1. $C_4 \leftarrow C_4 + C_5$
2. $C_3 \leftarrow C_3 + C_4$
3. $C_2 \leftarrow C_2 + C_3$
4. $C_1 \leftarrow C_1 + C_2$

$$= (x+4y)(x-y)^4 \qquad \blacksquare$$

Invertible Matrices

Def An $n \times n$ matrix A is invertible if there exist an $n \times n$ matrix B such that

$$AB = I_n = BA$$

where $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{bmatrix}$ (the identity matrix).

<u>Thm</u> A matrix A is invertible if and only if $\det(A) \neq 0$.

<u>Ex</u> • $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible since $\det(A) = 0$.

• $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible since $\det(A) = 1$.

<u>Thm</u> A matrix A is invertible if and only if the only solution to the equation

$$A\vec{x} = \vec{0} \qquad (x = n\times n \text{ matrix})$$

is $\vec{x} = \vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

<u>Ex</u> Find all values of $\lambda \in \mathbb{R}$ such that

where $\vec{x} \neq 0$ $\quad A\vec{x} = \lambda x$ and $\quad A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix}$.

<u>Solution</u> $A\vec{x} = \lambda\vec{x} \iff A\vec{x} - \lambda\vec{x} = \vec{0}$

$\iff A\vec{x} - \lambda I_3\vec{x} = \vec{0}$

$\iff (A - \lambda I_3)\vec{x} = \vec{0}$

Since $\vec{x} \neq 0$, by the Thm, $A - \lambda I_3$ is not an invertible matrix $\Rightarrow \det(A - \lambda I_3) = 0$

$$\iff \det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 2 \\ 0 & 1 & -\lambda \end{bmatrix} = 0$$

$$\iff (1-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = 0$$

$\iff (1-\lambda)\left[(1-\lambda)(-\lambda) - 2\right] = 0$

$\iff (1-\lambda)\left(-\lambda + \lambda^2 - 2\right) = 0$

$\iff (1-\lambda)(\lambda-2)(\lambda+1) = 0$

So the solutions to $A\vec{x} = \lambda\vec{x}$ are $\lambda = 1, 2, -1$ ∎

The values of λ are called Eigenvalues of A.

You will study this in linear algebra!