Determinant of a Matrix

\[ \text{Def: } \det \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \]

\[ \det \left( \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \right) = a \det \left( \begin{bmatrix} e & f \\ h & i \end{bmatrix} \right) - b \det \left( \begin{bmatrix} d & f \\ g & i \end{bmatrix} \right) + c \det \left( \begin{bmatrix} d & e \\ g & h \end{bmatrix} \right) \]

Motivation (geometric) Consider \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). A can be thought of as a map \( L_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), \( L_A(\hat{e}_1) = A\hat{e}_1 \), \( \hat{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \hat{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) be the standard basis for \( \mathbb{R}^2 \). We have

\[ L_A(\hat{e}_1) = \begin{bmatrix} a \\ c \end{bmatrix} \quad L_A(\hat{e}_2) = \begin{bmatrix} b \\ d \end{bmatrix} \]

Area of parallelogram: \( (a+b)(c+d) = 2bc - ac - bd = ac - bd \)

signed

Conclusion: \( \det(A) \) is the area of the parallelogram spanned by the columns of \( A \) (or rows).

In general: for an \( n \times n \) matrix \( B \), \( \det(B) \) the "volume" of the "parallelepiped" spanned by the columns.

Application: Three vectors \( a, b, c \in \mathbb{R}^3 \) are coplanar if and only if \( \det(\begin{bmatrix} a & b & c \end{bmatrix}) = 0 \) (since the parallelepiped is flat)
The Cross Product in $\mathbb{R}^3$

**Definition (Geometric)** Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, the cross product $\mathbf{u} \times \mathbf{v}$ is the vector $\mathbf{w}$ with the following properties:

1. perpendicular to both $\mathbf{u}$ and $\mathbf{v}$;
2. $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta$, $\theta =$ angle between them;
3. direction determined by the right hand rule.

**Formula** If $\mathbf{u} = (a, b, c)$, $\mathbf{v} = (x, y, z)$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix}
\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\
a & b & c \\
x & y & z
\end{vmatrix}$$

Then two vectors are parallel if and only if $|\mathbf{u} \times \mathbf{v}| = 0$ if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$

**Proof:** $\mathbf{u}, \mathbf{v}$ are parallel $\iff \theta = 0$ or $\pi$ $\iff \sin \theta = 0$ $\iff |\mathbf{u} \times \mathbf{v}| = 0$ $\iff \mathbf{u} \times \mathbf{v} = \mathbf{0}$.

**Application** 3 points are collinear if and only if $\mathbf{b} - \mathbf{a}$, $\mathbf{c} - \mathbf{a}$ are parallel if and only if $\mathbf{b} - \mathbf{a} \times \mathbf{c} - \mathbf{a} = \mathbf{0}$.

**Interesting Fact:** The cross product only exists in $\mathbb{R}^n$ if $n = 0, 1, 3$, or $7$ (Nontrivial to show).
The Equation of a Plane

- Let \( \mathbf{n} = (a, b, c) \) be a vector orthogonal to \( P \).
- Let \( \mathbf{r}_0 = (x_0, y_0, z_0) \) be a point in the plane.

Assume \( \mathbf{r} = (x, y, z) \) lies in \( P \). Then \( \mathbf{r} - \mathbf{r}_0 \) is parallel to \( P \).
So \( \mathbf{n} \) is orthogonal to \( \mathbf{r} - \mathbf{r}_0 \) \( \Rightarrow \)

\[
\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \quad \text{(eq. of a plane)}
\]

Expand: \( a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \)
Problem 1

(a) Show that the lines \( r_1(t) = (1, 1, 0) + t(1, -1, 2) \), \( r_2(s) = (2, 0, 2) + s(-1, 1, 0) \) intersect.

(b) Find a plane containing both lines.

Solution (a) Solve the system of equations:

\[
(1 + t, 1 - t, 2t) = r_1(t) = r_2(s) = (2 - s, s, 2)
\]

\[
\Rightarrow \\
(1) \quad 1 + t = 2 - s \\
(2) \quad 1 - t = s \\
(3) \quad 2t = 2
\]

By (3), \( t = 1 \)

By (1), \( 1 + 1 = 2 - s \)

\[\Rightarrow s = 0\]

And (2) is consistent when \( s = 0, t = 1 \)

So the lines do intersect when \( s = 0, t = 1 \). So the point of intersection is

\[r_1(1) = (2, 0, 2)\]

(b)

A point in the plane is \( (2, 0, 2) \). A normal vector is

\[
h = (1, -1, 2) \times (-1, 1, 0)
\]

\[
= \left| \begin{array}{ccc}
\hat{i} & \hat{j} & \hat{k} \\
1 & -1 & 2 \\
-1 & 1 & 0 \\
\end{array} \right|
\]

\[
= \hat{i} \left| \begin{array}{cc}
-1 & 2 \\
1 & 0 \\
\end{array} \right| - \hat{j} \left| \begin{array}{cc}
1 & 2 \\
-1 & 0 \\
\end{array} \right| + \hat{k} \left| \begin{array}{cc}
1 & -1 \\
-1 & 1 \\
\end{array} \right|
\]

\[
= (2, -2, 0)
\]

So an eq of the plane is:

\[(-2, -2, 0) \cdot ((x, y, z) - (2, 0, 2)) = 0\]
Problem 2

(1) Parameterize the line of intersection of the planes \( \frac{x+y+z}{3} = 1 \) and \( x-y+z = 1 \).

(2) Find the equation of the plane orthogonal to this line that contains \((1,1,1)\).

Solution

\[ n_1 = (1,1,1) \]
\[ n_2 = (1,-1,1) \]

\( n = n_1 \times n_2 \)

\[ n = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i}(-1) - \hat{j}(1) + \hat{k}(2) = (2,0,-2) \]

So the line is

\[ l(t) = (1,0,0) + t(2,0,-2) \]

(2) The normal vector for the plane is \((2,0,-2)\) and a point in the plane is \((1,1,1)\), so the eq is

\[ (2,0,-2) \cdot ((x,y,z) - (1,1,1)) = 0 \]
Problem 3

Eq. of the plane containing \( \mathbf{r}(t) = (-1, 1, 2) + t(3, 2, 4) \) that is perpendicular to the plane \( 2x + y - 3z = -4 \).