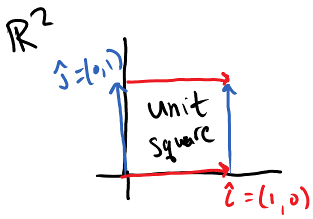
Determinant of a Matrix

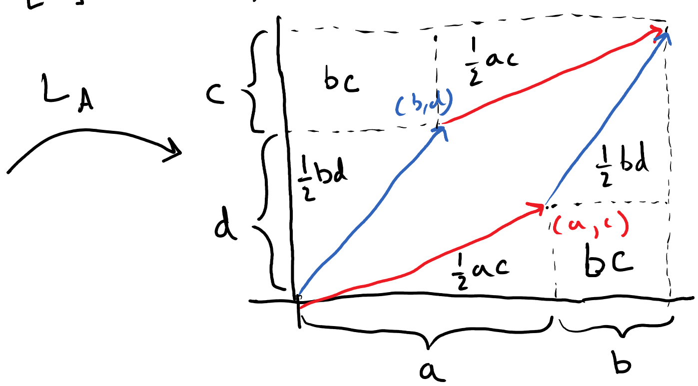
Def $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\det\left(\begin{bmatrix} \overset{+}{a} & \overset{-}{b} & \overset{+}{c} \\ d & e & f \\ h & i & j \end{bmatrix}\right) = a\begin{vmatrix} e & f \\ i & j \end{vmatrix} - b\begin{vmatrix} d & f \\ h & j \end{vmatrix} + c\begin{vmatrix} d & e \\ h & i \end{vmatrix}$$

Motivation (Geomitric) Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. A can be thought of as a map $L_A : \mathbb{R}^{2} \to \mathbb{R}^{2}$, $L_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix}$. Let $\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ be the standard basis for $\mathbb{R}^2$. We have

$$L_A\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a \\ c \end{bmatrix} \qquad L_A\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} b \\ d \end{bmatrix}$$





Area of parallelogram: $(a+b)(c+d) - 2bc - ac - bd = ac - bd$
$= \det(A)$

Conclusion: $\det(A)$ is the signed area of the parallelogram spanned by the columns of A (or rows).

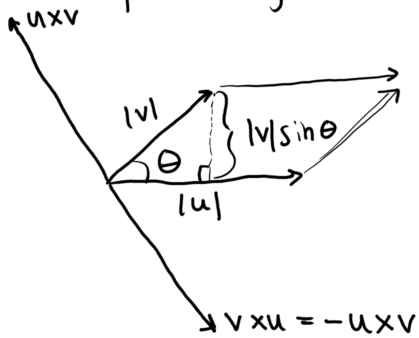
In general: for an $n \times n$ matrix B, $\det(B)$ the "volume" of the "parallelepiped" spanned by the columns

Application Three vectors $a, b, c \in \mathbb{R}^3$ are coplanar if and only $\det([a\ b\ c]) = 0$ (since the parallelepiped is flat)

The Cross Product in $\mathbb{R}^3$

Chapter 11.3

Def (Geometric) Let $u, v \in \mathbb{R}^3$, the cross product $u \times v$ is the vector w/ the following properties
(1) perpindicular to both u and v;
(2) $|u \times v| = |u||v| \sin\theta$, θ = angle between them
(3) direction determined by the right hand rule.



$\leadsto$ so $|u \times v|$ = area of the parallelogram spanned by u, v

Formula If $u = (a, b, c)$, $v = (x, y, z)$, then

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ x & y & z \end{vmatrix}$$

Thm Two vectors are parallel if and only if $|u \times v| = 0$
if and only if $u \times v = \vec{0}$

Pf: u, v are parallel $\Leftrightarrow$ $\theta = 0$ or π
$\Leftrightarrow$ $\sin\theta = 0$ $\Leftrightarrow$ $|u \times v| = 0$
$\Leftrightarrow$ $u \times v = \vec{0}$.

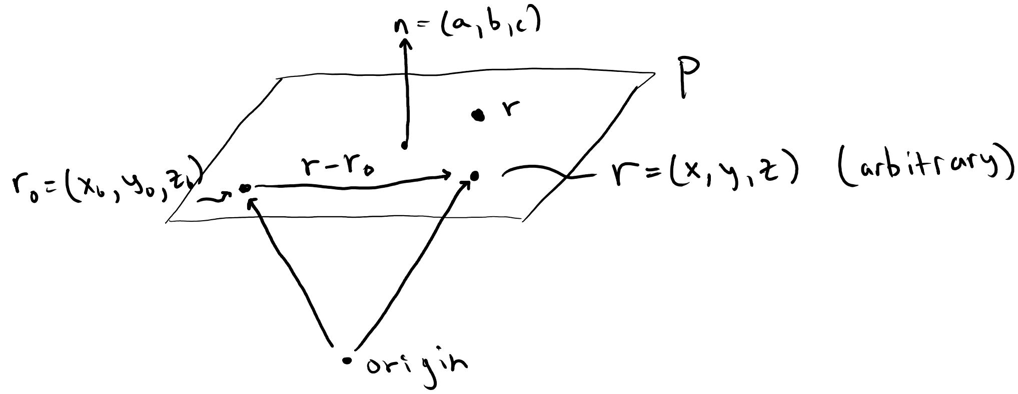
Application 3 points ($a, b, c \in \mathbb{R}^3$) are collinear if and only if

$\overrightarrow{b-a}$, $\overrightarrow{c-a}$ are parallel if and only if $\overrightarrow{b-a} \times \overrightarrow{c-a} = \vec{0}$.

Interesting Fact: The cross product only exists in $\mathbb{R}^n$ if $n = 0, 1, 3$, or 7. (Nontrivial to show)

The Equation of a Plane

Chapter 11.3



- Let $n = (a, b, c)$ be a vector orthogonal to P.
- Let $r_0 = (x_0, y_0, z_0)$ be a point in the plane.

Assume $r = (x, y, z)$ lies in P. Then $r - r_0$ is parallel to P.
So n is orthogonal to $r - r_0$ $\Longrightarrow$

$$\boxed{n \cdot (r - r_0) = 0} \quad \text{(eq. of a plane)}$$

Expand: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

■

Problem 1 | Chapter 11.3

(a) Show that the lines $r_1(t) = (1,1,0) + t(1,-1,2)$, $r_2(s) = (2,0,2) + s(-1,1,0)$ intersect.
(b) Find a plane containing both lines.

Solution (a) Solve the system of equations:

$$(1+t, 1-t, 2t) = r_1(t) = r_2(s) = (2-s, s, 2)$$

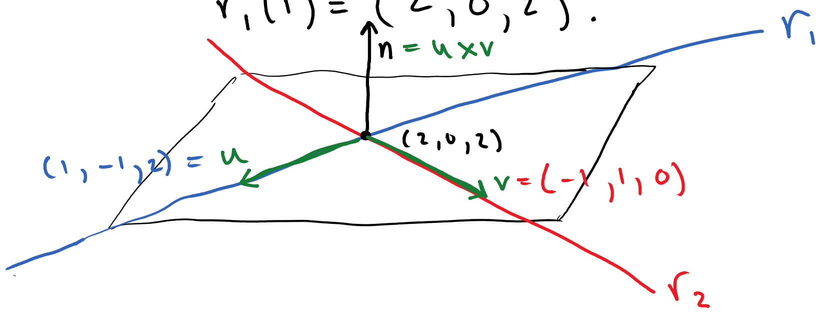
$\Rightarrow$ (1) $1+t = 2-s$
(2) $1-t = s$
(3) $2t = 2$

By (3), $t=1$
By (1), $1+1 = 2-s$
$\Rightarrow s = 0$
And (2) is consistent when $s=0, t=1$

So the lines do intersect when $s=0, t=1$. So the point of intersection is

$$r_1(1) = (2,0,2).$$

(b)



A point in the plane is $(2,0,2)$. A normal vector is

$$n = (1,-1,2) \times (-1,1,0)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$$

$$= (-2, -2, 0)$$

So an eq of the plane is

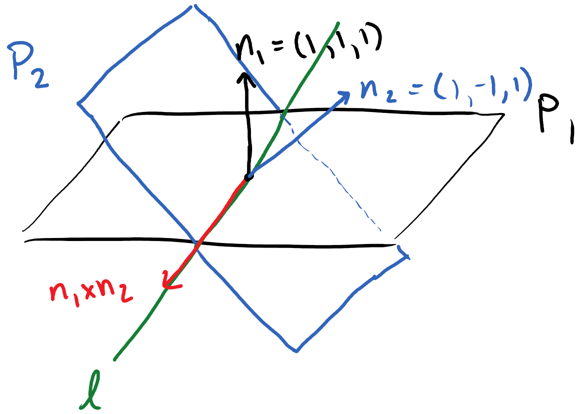
$$\boxed{(-2,-2,0) \cdot ((x,y,z) - (2,0,2)) = 0}$$

Problem 2 — Chapter 11.3

(1) Parameterize the line of intersection of the planes $\underbrace{x+y+z=1}_{P_1}$ & $\underbrace{x-y+z=1}_{P_2}$

(2) Find eq. of the plane orthogonal to this line that contains $(1,1,1)$

Solution



(1) A point on l is contained in P_1 and P_2, say

$$(1,0,0)$$

The direction of l is $n = n_1 \times n_2$

$$n = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$
$$= (2, 0, -2)$$

So the line is

$$l(t) = (1,0,0) + t\,(2,0,-2)$$

(2) The normal vector for the plane is $(2,0,-2)$ and a point in the plane is $(1,1,1)$, so the eq is

$$\boxed{(2,0,-2)\cdot\big((x,y,z)-(1,1,1)\big) = 0}$$

Problem 3

Chapter 11.3

Eq. of the plane containing $\ell(t) = (-1, 1, 2) + t(3, 2, 4)$ that is perpindicular to the plane $2x + y - 3z = -4$.