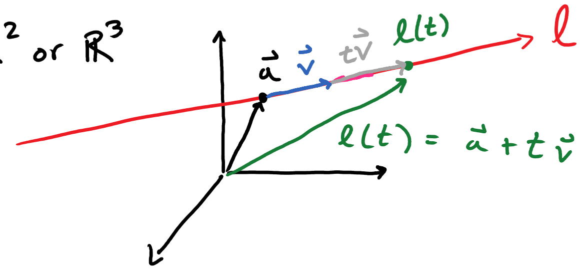
<u>Equation of a line</u> <u>Chapter 11.1</u>

$\mathbb{R}^2$ or $\mathbb{R}^3$



- Let a be any point on the line
- Let $\vec{v}$ be any vector parallel to ℓ.

Then eq. of the line determined by a and v is

$$\ell(t) = \vec{a} + t\vec{v}$$

<u>Comparing line</u> Given $r(t) = a + tv$ and $s(t) = b + tw$, how can you determine if they represent the same line?

<u>A</u>: Only if a lies on $s(t)$ $(s(t_0) = a)$ and v and w should be parallel $(v = cw$ for some $c \in \mathbb{R})$.

<u>Ex</u> Consider $r(t) = \underbrace{(0,-2,-3)} + t(-2,-4,-6)$ and $s(t) = (1,0,0) + t(1,2,3)$.

Note that $-2(1,2,3) = (-2,-4,-6)$ so the lines have the same direction. Also, $s(-1) = (0,-2,-3)$ so $(0,-2,-3)$ lies on s.

So the lines are the same.

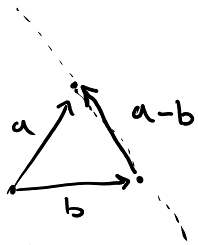
<u>Note</u>: There are infinitely many ways to parameterize the same line.

Problem 1 Chapter 11.1

Are the points $(2,1,3)$, $(2,3,4)$, and $(2,-3,1)$ collinear?

Solution

(1) Find the equation of the line containing $(2,1,3)$ and $(2,3,4)$.



A vector parallel to the line is $v = (2,1,3) - (2,3,4)$
$= (0,-2,-1)$

Then the equation of the line is

$$\ell(t) = (2,1,3) + t(0,-2,-1)$$
$$= (2, 1-2t, 3-t)$$

(2) Check if $(2,-3,1)$ lies on $\ell(t)$. We solve the equation

$$(2,-3,1) = \ell(t) = (2, 1-2t, 3-t)$$

$$\begin{cases} 2 = 2 \\ -3 = 1-2t \Rightarrow t = 2 \\ 1 = 3-t \Rightarrow t = 2 \end{cases}$$ So $\ell(2) = (2,-3,1)$

Which means the collinear! ■

Problem 2 **Chapter 11.1**

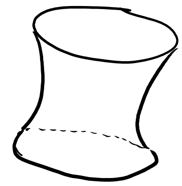
Find a line that lies inside the surface defined by the equation:

$$x^2 + y^2 - z^2 = 1$$

Solution

(1) Choose a nice point in the surface like $a = (0,1,0)$.

$x^2+y^2-z^2=1$ is called a hyperboloid of one-sheet



(2) Let $v = (a,b,c)$ and consider the line $\ell(t) = a + tv$

$$= (\underbrace{ta}_{x(t)}, \underbrace{1+tb}_{y(t)}, \underbrace{tc}_{z(t)})$$

The line $\ell(t)$ lies in the surface if:

$$(x(t))^2 + (y(t))^2 - (z(t))^2 = 1$$

for all $t \in \mathbb{R}$. That is

$$(ta)^2 + (1+tb)^2 - (tc)^2 = 1$$

Set $a = c$, then the equation becomes

$$(1+tb)^2 = 1$$

If $b = 0$, then the equation holds for all $t \in \mathbb{R}$. So

$$v = (a,0,a) \quad \text{for some } a \in \mathbb{R}$$

So an equation of a line lying in the surface is

$$\ell(t) = (0,1,0) + t(a,0,a)$$
$$= (at, 1, at)$$

Dot product

Chapter 11.2

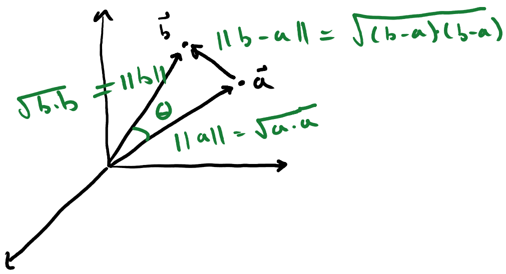
Def The dot product of two vectors $\vec{a} = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ is the real number

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3 \in \mathbb{R}.$$

Q: What does the dot product tell us about a and b.
A: The dot product is a measure of the angle between a and b.

(More generally, inner products are used to define angles)

Claim: $a \cdot b = \|a\| \cdot \|b\| \cos\theta$, θ = angle betwee n a and b.



By Law of Cosines, $\|b-a\|^2 = \|b\|^2 + \|a\|^2 - 2\|a\|\|b\|\cos\theta$.

Using the identity $\|v\|^2 = v \cdot v$, we get

$$\|b-a\|^2 = (b-a)\cdot(b-a) = b \cdot b - 2a \cdot b + a \cdot a$$

Then the equation becomes

$$\cancel{b \cdot b} - 2a \cdot b + \cancel{a \cdot a} = \cancel{b \cdot b} + \cancel{a \cdot a} - 2\|a\|\|b\|\cos\theta$$

$$\Rightarrow \quad -2a \cdot b = -2\|a\|\|b\|\cos\theta.$$

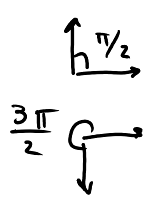
So the identity $a \cdot b = \|a\|\|b\|\cos\theta$ is the law of cosines in disguise.

Thm $a \cdot b = 0$ if and only if $a \perp b$

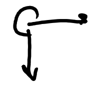
Proof $a \cdot b = 0 \iff \cos\theta = 0$, $0 \leq \theta \leq 2\pi$

$\iff \theta = \pi/2, \ 3\pi/2$

$\iff a \perp b$



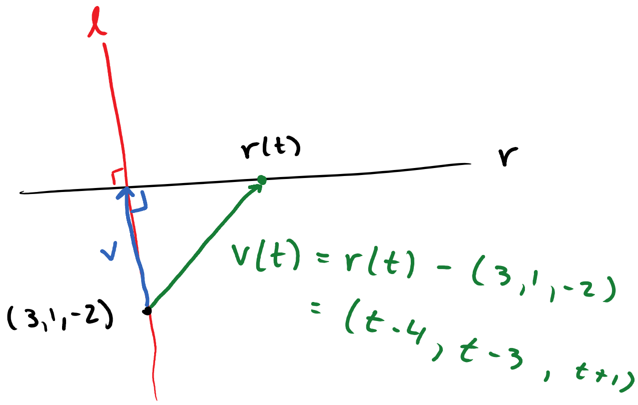
$\iff a \perp b$

$\frac{\pi}{2}$ 

Problem 3

Find a line through $(3,1,-2)$ that intersects and is perpendicular to the line $r(t) = (t-1, t-2, t-1)$.

Solution



- A point on ℓ is $(3,1,-2)$
- Consider the vector $v(t) = (t-4, t-3, t+1)$. The direction of r is the vector $(1,1,1)$. $v(t)$ and r are perpendicular if and only if

$$\begin{aligned} 0 &= v(t) \cdot (1,1,1) \\ &= t-4+t-3+t+1 \\ &= 3t-6 \end{aligned}$$

So $t=2$. That is, $v = v(2) = (-2,-1,3)$. Then

$$\boxed{\ell(t) = (3,1,-2) + t(-2,-1,3)}$$

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