1. (20 points) Let $\vec{u} = (-1, 1, 0)$ and $\vec{v} = (2, -1, 2)$. Find the following:

(a) $\vec{u} \cdot \vec{v}$ $\boxed{-3}$

(b) $\vec{u} \times \vec{v}$ $\boxed{(2,2,-1)}$

(c) $\|\vec{u}\|$ and $\|\vec{v}\|$ $\|\vec{u}\| = \boxed{\sqrt{2}}$ $\|\vec{v}\| = \boxed{3}$

(d) Give the cosine of the angle θ between the vectors $\vec{u}$ and $\vec{v}$.

$\cos\theta = \boxed{-\frac{\sqrt{2}}{2}}$

(e) Find the area of the parallelogram spanned by the vectors $\vec{u}$ and $\vec{v}$.

$\boxed{3}$

1. a) $u \cdot v = (-1,1,0)\cdot(2,-1,2) = -2-1+0 = -3$

b) $\begin{vmatrix} i & j & {}^{+}k \\ -1 & 1 & {}^{-}0 \\ 2 & -1 & {}^{+}2 \end{vmatrix} = k\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} + 2\begin{vmatrix} i & j \\ -1 & 1 \end{vmatrix}$

$= k(1-2) + 2(i+j) = (2,2,-1)$

c) $\|u\| = \sqrt{(-1)^2+1^2+0^2} = \sqrt{2}$

$\|v\| = \sqrt{2^2+(-1)^2+2^2} = \sqrt{9} = 3$

d) $u\cdot v = \|u\|\|v\|\cos\theta \Rightarrow \theta = \cos^{-1}\left(\frac{u\cdot v}{\|u\|\|v\|}\right)$

$= \cos^{-1}\left(\frac{-3}{\sqrt{2}\cdot 3}\right)$

$= \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$

e) $\|u\times v\| = \sqrt{2^2+2^2+1} = \sqrt{9} = 3$

2. (5 points) Compute the determinant of the 3×3 matrix:

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 0 & -2 \\ 4 & -2 & 6 \end{bmatrix}$$

(A) -36 (B) -16 (C) 0 (D) 16 (E) 36 Answer (Letter): $\boxed{C}$

2. $\begin{vmatrix} 2 & -1 & 3 \\ {}^{-}3 & {}^{+}0 & {}^{-}-2 \\ 4 & -2 & 6 \end{vmatrix} = 3\begin{vmatrix} -1 & 3 \\ -2 & 6 \end{vmatrix} - (-2)\begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix}$

$= 3(-6-(-6)) + 2(-4-(-4))$

$= 0.$

3. (12 points) Let $\mathcal{P}$ be the plane that contains the point $Q(1,2,4)$ and contains the line $\mathbf{l}(t) = (1,-3,5) + t(2,1,-4)$

(a) Give two non-parallel vectors $\vec{v_1}, \vec{v_2}$ that lie in the plane $\mathcal{P}$.

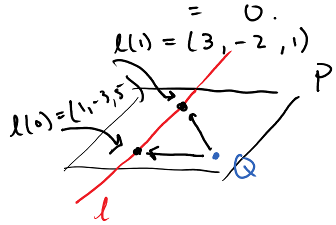
$\vec{v_1} = \boxed{(0,5,-1)}$ $\vec{v_2} = \boxed{(-2,4,3)}$

(b) Give a normal vector $\vec{n}$ to the plane $\mathcal{P}$. $\vec{n} = \boxed{(19,2,10)}$

(c) Give an equation of the plane $\mathcal{P}$ in the form $ax + by + cz = d$.

$\boxed{19x + 2y + 10z = 63}$

3.



Check: Does Q lie on l?

$(1,2,4) = (1+2t, -3+t, 5-4t)$

$\Rightarrow 1 = 1+2t \Rightarrow t = 0$

$\Rightarrow 2 = -3+t = -3+0$. Nonsense

Q does not lie on l.

(a) $v_1 = Q - l(0) = (1,2,4) - (1,-3,5)$
$= (0,5,-1)$

$v_2 = Q - l(1) = (1,2,4) - (3,-2,1)$
$= (-2,4,3)$

(b) $n = |v_1 \times v_2| = \begin{vmatrix} i^{+} & j & k \\ 0^{-} & 5 & -1 \\ -2^{+} & 4 & 3 \end{vmatrix} = i\begin{vmatrix} 5 & -1 \\ 4 & 3 \end{vmatrix} - 2\begin{vmatrix} j & k \\ 5 & -1 \end{vmatrix}$

$= i(15+4) - 2(-j-5k)$

$= (19, 2, 10)$

(c) $(19,2,10)\cdot((x,y,z) - (1,2,4)) = 0$

$\Rightarrow 19x - 19 + 2y - 4 + 10z - 40 = 0$

$\Rightarrow 19x + 2y + 10z = 63$

4. (10 points) Consider the point $P(3, \frac{\pi}{2}, \frac{\pi}{2})$ in spherical coordinates. Give the rectangular and cylindrical coordinates of P.

Rectangular coordinates of P: $\boxed{(0,3,0)}$

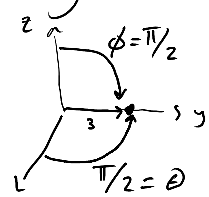
Cylindrical coordinates of P: $\boxed{(3,\pi/2,0)}$

4. $(\rho, \theta, \phi) = (3, \pi/2, \pi/2)$

$\left.\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}\right\}$ rectangular

Rectangular: $(x,y,z) = (0,3,0)$

cylindrical: $(r,\theta,z) = (\rho,\theta,z)$
$= (3, \pi/2, 0)$



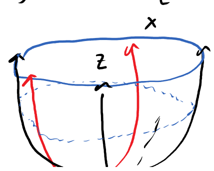
5. (5 points) The quadratic surface given by $z - 2x^2 - y^2 = 1$ is

(A) an ellipsoid (B) a hyperboloid of one sheet (C) a hyperboloid of two sheets
(D) a cone (E) an elliptic paraboloid

Answer (Letter): $\boxed{E}$

5. Set $x = c$. $\Rightarrow$ $z = \underbrace{(1+2c^2)}_{\geq 1} + y^2$

Parabolas



Set $y = c$ $\Rightarrow$ $z = (1+c^2) + 2x^2$

(A) an ellipsoid (B) a hyperboloid of one sheet (C) a hyperboloid of two sheets
(D) a cone (E) an elliptic paraboloid

Answer (Letter): $\boxed{E}$

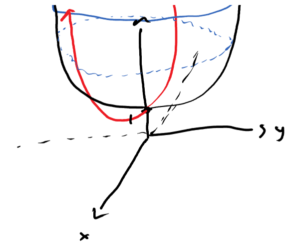
Parabolas ≥ 1

Set $y=c \Rightarrow z = \underbrace{(1+c^2)}_{\geq 1} + 2x^2$

Parabolas

Set $z=c \Rightarrow (c-1) = 2x^2 + y^2$

Ellipses if $c>1$ a point if $c=1$
empty if $c<1$



6. (10 points) Consider the path $\mathbf{c}(t) = (4\sin(t), 2\cos(t), 2\sin t + \cos t)$. Find the following

(a) The tangent (velocity) vector to this path at $t=0$: $\boxed{(4,0,2)}$

(b) The acceleration vector to this path at $t=0$: $\boxed{(0,-2,-1)}$

(c) The parametric equation for the tangent line to this path $\mathbf{c}(t)$ at $t=0$:

$x = \boxed{4t}$ $y = \boxed{2}$ $z = \boxed{1+2t}$

6. (a) $c'(t) = (4\cos t, -2\sin t, 2\cos t - \sin t)$

$\Rightarrow c'(0) = (4, 0, 2)$

(b) $c''(t) = (-4\sin t, -2\cos t, -2\sin t - \cos t)$

$c''(0) = (0, -2, -1)$

(c) $C(0) = (0,2,1)$ is a point in the line

$V(0) = c'(0) = (4,0,2)$ is the direction of the line.

$l(t) = c(0) + t\,v(0)$
$= (0,2,1) + t(4,0,2)$
$= (4t, 2, 1+2t)$

7. (10 points) Consider the path $\mathbf{c}(t) = (2t, \ln(t), t^2)$ where $1 \leq t \leq 2$.

(a) Set up an integral for the length of the path $\mathbf{c}(t)$ where $1 \leq t \leq 2$.

$\boxed{\int_1^2 \sqrt{2^2 + \frac{1}{t^2} + 4t^2}\, dt}$

(b) Find the arc length L of the path $\mathbf{c}(t)$ where $1 \leq t \leq 2$.

$L = \boxed{3 + \ln 2}$

7. (a) $\int_1^2 \|c'(t)\|\,dt = \int_1^2 \sqrt{2^2 + \frac{1}{t^2} + 4t^2}\,dt$

(b) $\int_1^2 \sqrt{2^2 + \frac{1}{t^2} + 4t^2}\,dt = \int_1^2 \sqrt{(2t + \frac{1}{t})^2}\,dt$

$= \int_1^2 2t + \frac{1}{t}\,dt$

$= \left[t^2 + \ln t\right]_1^2$

$= 4 + \ln 2 - 1$

$= 3 + \ln 2$

8. (5 points) Find the (x,y,z) coordinates of the point P where the line $\mathbf{l}(t) = (x,y,z) = (3-t, 2+t, 5t)$ intersects the plane $x - y + 2z = 9$.

$P = \boxed{(2,3,5)}$

8. Solve for t:

$(3-t) - (2+t) + 2(5t) = 9$

$\Rightarrow 8t = 9 - 1 \Rightarrow t = 1$

$\Rightarrow l(1) = (3-1, 2+1, 5\cdot 1) = (2,3,5)$

9. (12 points) Let $f(x,y) = x^4 y - y^2$.

(a) Find $\frac{\partial f}{\partial x}(x,y)$ $\boxed{4x^3 y}$

(b) Find $\frac{\partial f}{\partial y}(x,y)$ $\boxed{x^4 - 2y}$

(c) Find the equation of the tangent plane to $z = f(x,y)$ at the point $(1,1)$.

$\boxed{z = 4x - y - 3}$

9. (a) $f_x = 4x^3 \cdot y - 0 = 4x^3 y$

(b) $f_y = x^4 - 2y$

(c) Eq of the tangent plane is:

$z - f(x_0,y_0) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$f(1,1) = 0$ $f_x(1,1) = 4$ $f_y(1,1) = -1$

So the eq. is $z = 4(x-1) + (-1)(y-1)$
$= 4x - y - 3$

10. (6 points) Find the following limits, if possible. If a limit does not exist, write DNE.

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} =$ $\square$

xy $\square$

10. (a) Use polar coordinates, $x = r\cos\theta$ $y = r\sin\theta$:

$\lim \quad x^2y^2 \qquad \lim \quad r^2\cos^2\theta\, r^2\sin^2\theta$

10. (6 points) Find the following limits, if possible. If a limit does not exist, write DNE.

(a) $\displaystyle\lim_{(x,y)\longrightarrow(0,0)} \frac{x^2y^2}{x^2+y^2} =$ ☐

(b) $\displaystyle\lim_{(x,y)\longrightarrow(0,0)} \frac{xy}{x^2+y^2} =$ ☐

10. (a) Use polar coordinates, $x = r\cos\theta$ $y = r\sin\theta$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\cos^2\theta\, r^2\sin^2\theta}{r^2}$$

$$= \lim_{r\to 0} r^2\sin^2\theta\cos^2\theta = 0$$

(b) Two ways: (1) use polar

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\sin\theta\cos\theta}{r^2}$$

$$= \lim_{r\to 0} \sin\theta\cos\theta$$

Fix $\theta = 0$

$= \sin 0 \cos\theta$

$= 0$

Fix $\theta = \pi/4$

$= \sin\pi/4 \cos\pi/4$

$= \left(\frac{\sqrt{2}}{2}\right)^2 \neq 0$

So the limit does not exist.

(2) Find two different paths approaching $(0,0)$ in the xy-plane that give different values in the limit.

Path 1: Approach along the x-axis, $c_1(t) = (t, 0)$

Then the limit is:

$$\lim_{t\to 0} \frac{x_1(t)\,y_1(t)}{x_1(t)^2 + y_1(t)^2} = \lim_{t\to 0} \frac{t\cdot 0}{t^2+0} = 0$$

Path 2: The line $y = x$, $c_2(t) = (x_2(t), y_2(t)) = (t,t)$

$$\Rightarrow \lim_{t\to 0} \frac{x_2(t)\,y_2(t)}{x_2(t)^2 + y_2(t)^2} = \lim_{t\to 0} \frac{t^2}{t^2+t^2} = \frac{1}{2}$$

So limit DNE.

11. (5 points) Give the set of boundary points of the set $A = \{(x,y) \in \mathbb{R}^2 \mid 0 < x^2+y^2 \leq 9\}$

(A) A has no boundary points.

(B) $\{(0,0)\}$

(C) $\{0,3\}$

(D) $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 9\}$

(E) $\{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 9\}$

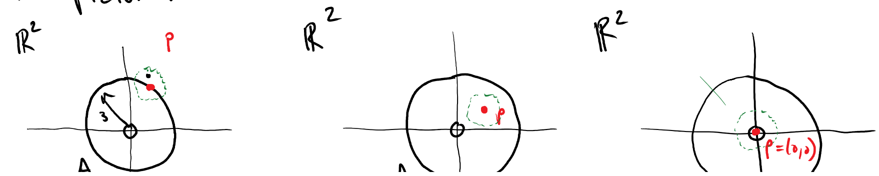
$<$

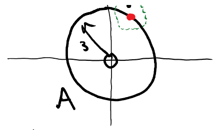
Answer (Letter): E

<u>Def</u> A point $p = (x,y)$ is a <u>boundary point</u> of $A \subseteq \mathbb{R}^2$ if for any $r > 0$, the open ball of radius r, centered at p contains atleast one point in A and one point not in A.

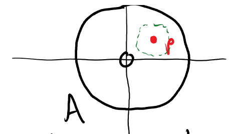
Open ball: $B_r(p) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < r^2\}$

The picture:

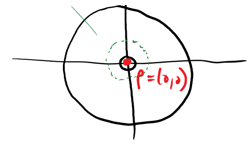




Points on the edge of the circle are boundary pts.



Points not on the edge (except (0,0)) are not boundary points



p=(0,0) is a boundary pt

7. (15 points) In this problem, let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined via $f(x, y) = (\underbrace{x^2y - x}_{f_1}, \underbrace{x^3 + y^3}_{f_2})$.
Find the matrix of partials $Df(x, y)$

$$\boxed{\begin{bmatrix} 2xy-1 & x^2 \\ 3x^2 & 3y^2 \end{bmatrix}}$$

7. Recall, the derivative of a function $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the 2×2 matrix:

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

So $$Df = \begin{bmatrix} 2xy-1 & x^2 \\ 3x^2 & 3y^2 \end{bmatrix}$$