# Strategies for Computing Limits



① Continuity

a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2+2}$ b) $\displaystyle\lim_{(x,y,z)\to(1,1,1)} \frac{x+y+z}{e^{xyz}}$

Def A function $f$ is continuous at $x_0$ if

$$\lim_{x\to x_0} f(x) = f(x_0)$$

a) $$\lim_{(x,y)\to(0,0)} \underbrace{\frac{xy}{x^2+y^2+2}}_{f(x,y)} = f(0,0) = 0$$

$f(x,y)$ is continuous since the numerator $xy$ is continuous (it's a polynomial) and the denominator is continuous and $x^2+y^2+2>0$. ■

b) $$\lim_{(x,y,z)\to(1,1,1)} \frac{x+y+z}{e^{xyz}} = \frac{3}{e}$$

The numerator is a polynomial and the denominator is a composition of continuous functions, and $e^{xyz}>0$ for all $(x,y,z)\in\mathbb{R}^3$.

■

In fact, both function are continuous on all of $\mathbb{R}^2$.

We need another strategy to compute limits where a function is not continuous.

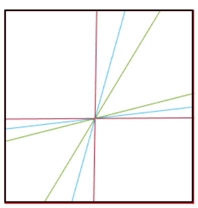
② Proving that a limit does not exist

a) $\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$ b) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

If the limit $\lim_{(x,y)\to(a,b)} f(x,y)$ exists, then the value of the limit should be the same as we approach $(a,b)$ along any path in the $x,y$-plane. So, to prove a limit does not exist, its sufficient to find two different paths that approach $(a,b)$ but give different values in the limit.

a) $\lim_{(x,y)\to(0,0)} \underbrace{\frac{(x-y)^2}{x^2+y^2}}_{f(x,y)}$

Solution. First, $f$ is undefined at $(0,0)$ and hence not continuous. One way to see that the limit may not exist: plot the level curves $f(x,y)=k$ for several values of $k$.



Each color is a path through $(0,0)$ at a distinct elevation. This suggests that the limit does not exist!

To prove the limit does not exist, first consider the line

$$c(t) = (t,t).$$

Take the limit along $c$:

$$\lim_{t\to 0} \frac{(t-t)^2}{t^2+t^2} = 0$$

Next, consider the line $r(t) = (t,0)$. Take the limit

$$\lim_{t\to 0} \frac{(t-0)^2}{t^2+0^2} = 1.$$

$$\lim_{t \to 0} \frac{(t-0)}{t^2+0^2} = 1 .$$

this proves the limit does not exist. $\blacksquare$

b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

Solution Consider the parabola $p(t) = (t^2, t)$. Take the limit

$$\lim_{t\to 0} \frac{t^2 \cdot t^2}{(t^2)^2 + t^4} = \lim_{t \to 0} \frac{t^4}{2t^4} = \frac{1}{2} .$$

Consider the line $c(t) = (t, t)$. The limit is

$$\lim_{t\to 0} \frac{t^3}{t^2+t^4} = \lim_{t\to 0} \frac{t^3}{t^2(1+t^2)} = \lim_{t\to 0} \frac{t}{1+t^2} = 0 .$$

This proves that the limit does not exist. $\blacksquare$

③ Change of coordinates (polar or spherical)

a) $\lim_{(x,y)\to(0,0)} (3x^2+3y^2)\ln(x^2+y^2)$  b) $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

c) $\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$  d) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

Polar Set $x = r\cos\theta$ and $y = r\sin\theta$

Spherical Set $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

a) $\lim_{(x,y)\to(0,0)} (3x^2+3y^2)\ln(x^2+y^2)$

Solution. Use polar coordinates Note that

$$\lim_{(x,y)\to(0,0)} r(x,y) = 0.$$

then we have

$$\lim_{r\to 0} 3r^2\ln(r^2) = \lim_{r\to 0} \frac{\ln(r^2)}{1/3r^2} \quad \left(\text{indeterminate form!}\right)$$

$$\overset{L'H}{=} \lim_{r\to 0} \frac{2r/r^2}{-1/6r^3}$$

$$= \lim_{r\to 0} -\frac{1}{3}r^2$$

$$= 0.$$

b) $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

Solution. Use polar coordinates:

$$\lim_{r\to 0} \frac{3r^3\cos^2\theta\sin\theta}{r^2} = \lim_{r\to 0} 3r\cos^2\theta\sin\theta$$

$$\lim_{r\to 0} \frac{\qquad}{r^2} = \lim_{r\to 0} 3r\cos^2\theta\sin\theta$$

Use squeeze theorem: note that

$$-1 \le \cos^2\theta\sin\theta \le 1 \qquad (\text{for all } \theta)$$

$$\Rightarrow \qquad -3r \le 3r\cos^2\theta\sin\theta \le 3r.$$

Since $\lim_{r\to 0} -3r = 0 = \lim_{r\to 0} 3r$ so also

$$\lim_{r\to 0} 3r\cos^2\theta\sin\theta = 0$$

∎

# Strategies for Computing Limits

④ One additional strategy

a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{y}$ b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\cos(xyz)-1}{x^2y^2z^2}$

We need the following result:

<u>Lemma</u> Suppose $f:\mathbb{R}^2\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ are functions such that

$$f(x,y) = g(xy) \quad \left(\text{for all } (x,y)\in\mathbb{R}^2\right)$$

If $(a,b)\in\mathbb{R}^2$ and $g$ is continuous at $ab\in\mathbb{R}$, then

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{t\to ab} g(t) = g(ab). \leftarrow$$

<u>Proof</u> Define a map $h:\mathbb{R}^2\to\mathbb{R}$ via $h(x,y)=xy$. This is continuous everywhere since it is a polynomial.

$$\begin{aligned}
\lim_{(x,y)\to(a,b)} f(x,y) &= \lim_{(x,y)\to(a,b)} g(xy) && \text{(by assumption)}\\
&= \lim_{(x,y)\to(a,b)} g(h(x,y))\\
&= g\left(\lim_{(x,y)\to(a,b)} h(x,y)\right) && (g \text{ is continuous at } ab)\\
&= g(h(a,b)) && (h \text{ is continuous})\\
&= g(ab)
\end{aligned}$$

∎

a) $\displaystyle\lim \; e^{xy}-1 = \lim \; x(e^{xy}-1)$ $\to f(x,y)$

a) $$\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{y} = \lim_{(x,y)\to(0,0)} \frac{x(e^{xy}-1)}{xy}$$

$f(x,y)$

$$= \underbrace{\left(\lim_{(x,y)\to(0,0)} x\right)}_{\text{exists}} \underbrace{\left(\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{xy}\right)}_{\text{need to show that this limit exist}}$$

Consider $g(t) = \begin{cases} \frac{e^t-1}{t} & , t \neq 0 \\ 1 & , t=0 \end{cases}$. Then

$g$ is continuous at $t=0$:

$$\lim_{t\to 0} g(t) = \lim_{t\to 0} \frac{e^t-1}{t} \overset{L'H}{=} \lim_{t\to 0} \frac{e^t}{1}$$

$$= 1 = g(0).$$

Moreover, $f(x,y) = g(xy)$. By the lemma

$$\lim_{(x,y)\to(0,0)} f(x,y) = g(0) = 1.$$

Thus,

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{y} = \left(\lim_{(x,y)\to(0,0)} x\right)\left(\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{xy}\right)$$

$$= 0 \cdot 1$$

$$= \quad 0 \cdot 1$$

$$= \quad 0 .$$

$\blacksquare$

b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\cos(xyz) - 1}{x^2y^2z^2}$

In a) we did a rigorous computation. In practice, we can think of this as the change of variables:

$$u = xyz .$$

We have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{\cos(xyz) - 1}{x^2y^2z^2} = \lim_{u\to 0} \frac{\cos(u) - 1}{u^2}$$

$$\overset{L'H}{=} \lim_{u\to 0} \frac{-\sin(u)}{2u}$$

$$\overset{L'H}{=} \lim_{u\to 0} \frac{-\cos u}{2} = -\frac{1}{2}$$

$\blacksquare$