Paths and Curves

$\boxed{\text{Chapter 12}}$

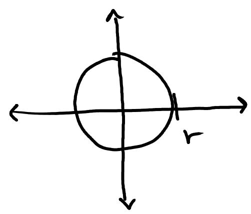
<u>Def</u> A path is a map $c: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$.

<u>Ex</u> (1) lines

(2) Circles: $c: [0, 2\pi] \longrightarrow \mathbb{R}^2$

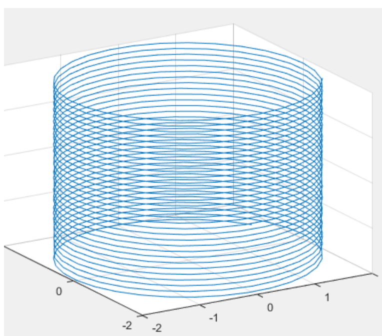
$$c(t) = (r\cos t, r\sin t)$$

Circle of radius r:



(3) Cylindrical Helix: $H: [0, 4\pi] \longrightarrow \mathbb{R}^3$ defined by

$$H(t) = (r\cos t, r\sin t, t)$$



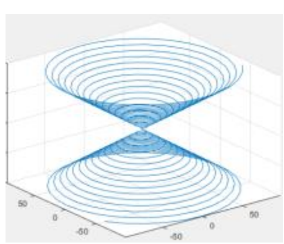
lies on the cylinder $x^2+y^2=r^2$

(4) Conic Helix: we want a helix that lies in the surface $x^2+y^2=z^2$. Suppose $c(t) = (r(t)\cos t, r(t)\sin t, t)$.

Assume $c(t)$ lies on the cone, then

$$r^2(t)\cos^2(t) + r^2(t)\sin^2(t) = t^2$$

$$\Rightarrow \quad r^2(t) = t^2 \Rightarrow r(t) = \pm t.$$



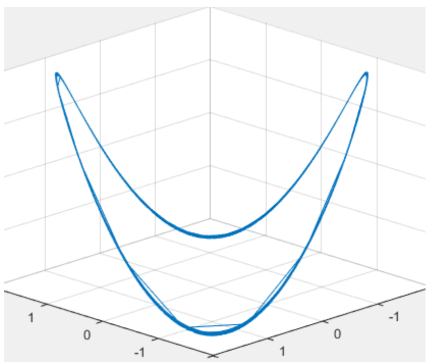
(5) Intersection of two surfaces:

(a) $x^2+y^2=4$ and $z=xy$

Suppose $c(t)=(x(t),y(t),z(t))$ lies on both surfaces.

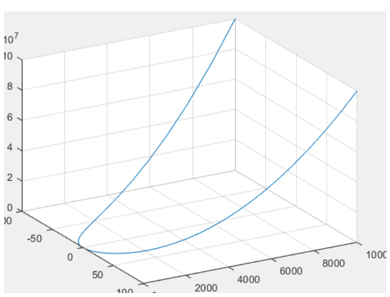
We can choose $x(t)=2\cos t$ $y(t)=2\sin t$ so that

$z(t)=x(t)y(t)=4\sin t\cos t=2\sin(2t)$.



(b) $z=4x^2+y^2$ and $y=x^2$

Set $x(t)=t$, then $y(t)=t^2$ and $z(t)=4t^2+t^4$



5) (From Hardcover Book, Marsden/Tromba, Vector Calculus, 6th ed., Section 2.4., # 6) Give a parameterization
(a) for the graph of $f(x)=x^2$
(b) for the ellipse given by $\frac{x^2}{9}+\frac{y^2}{25}=1$

(a) Set $x(t)=t$ $y(t)=t^2$

$c(t)=(t,t^2)$

(b) $x(t)=3\cos t$ $y(t)=5\sin t$

(b) $x(t) = 3\cos t \qquad y(t) = 5\sin t$

<u>Ex</u> Intersection of $z = x^2 - y^2$ and $x^2 + y^2 = 1$:

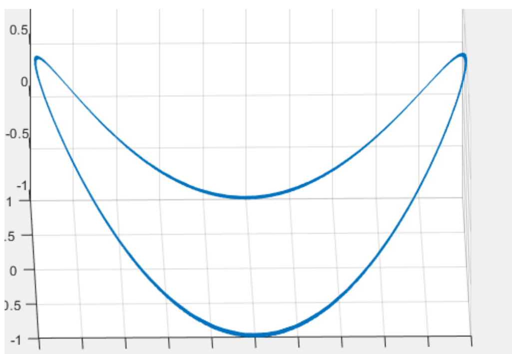
Find $c(t) = (x(t), y(t), z(t))$

(1) We must have $x^2(t) + y^2(t) = 1$ so we can take

$x(t) = \cos t \qquad y(t) = \sin t$ so that $z(t) = x^2(t) - y^2(t)$

$$= \cos^2 t - \sin^2 t$$

So $c(t) = (\cos t, \sin t, \cos(2t))$



(2) $z = x^2 - y^2$ and $x^2 + y^2 = 1$ (*)

Take $x(t) = t$. Using (*) we have $y^2 = 1 - x^2 = 1 - t^2$ so

that $z(t) = x^2 - y^2 = t^2 - (1 - t^2)$

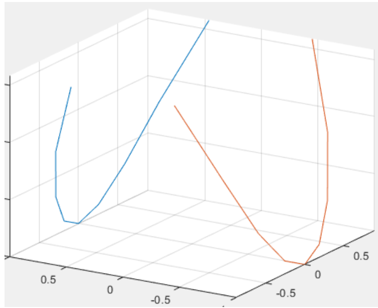
$$= 2t^2 + 1$$

Also using (*), $y = \pm\sqrt{1 - x^2} = \pm\sqrt{1 - t^2}$

So a parameterization is $c_1(t) = (t, \sqrt{1-t^2}, 2t^2 + 1) \quad 0 \le t \le 1$

$c_2(t) = (t, -\sqrt{1-t^2}, 2t^2 + 1)$

This didn't quite work!



Ex Intersection of the cone $z = +\sqrt{x^2+y^2}$ and the plane $z = 1+y$

$x(t) = t$ $\qquad 1+y = z = \sqrt{x^2+y^2}$

$\Rightarrow \quad (1+y)^2 = x^2+y^2$

$\Rightarrow \quad 1+2y+\cancel{y^2} = x^2+\cancel{y^2}$

$\Rightarrow \quad y(t) = \frac{x^2-1}{2} = \frac{t^2-1}{2}$

$\Rightarrow \quad z(t) = 1+y = 1+\frac{t^2-1}{2}$

$c(t) = \left(t, \frac{t^2-1}{2}, \frac{t^2+1}{2}\right)$