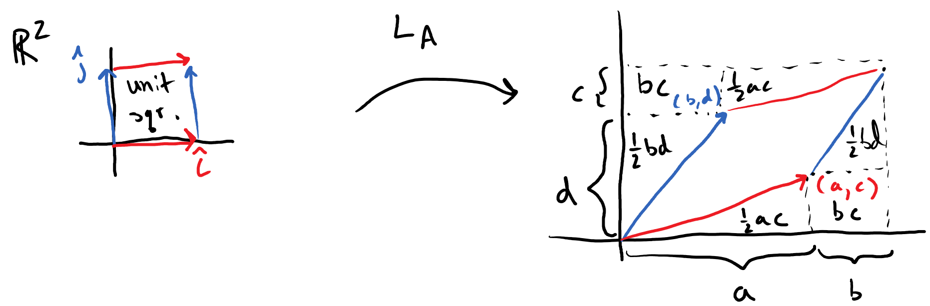
Determinant of a Matrix

Def $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\det\left(\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} a & b & c \\ {}^{-}d & {}^{+}e & {}^{-}f \\ {}^{+}g & {}^{-}h & {}^{+}i \end{bmatrix} \end{array}\right) = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Motivation (Geometry) Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We can view A as a map $L_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $L_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix}$. Let $\hat{\imath} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\hat{\jmath} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ be the standard basis of $\mathbb{R}^2$. Then

$$L_A(\hat{\imath}) = \begin{bmatrix} a \\ c \end{bmatrix} \quad \text{and} \quad L_A(\hat{\jmath}) = \begin{bmatrix} b \\ d \end{bmatrix}$$



Area of parallelogram: $(a+b)(c+d) - 2bc - ac - bd = ad - bc$
$= \det(A)$

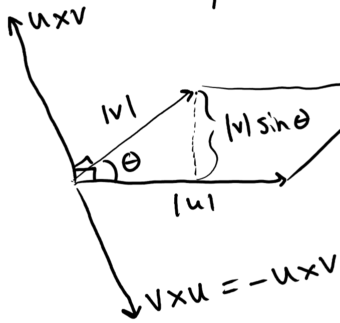
Conclusion $\operatorname{Det}(A) \overset{\text{signed area}}{=}$ area of parallelogram spanned by its colums (or rows). Also detects orientation of $L_A(\hat{\imath}), L_A(\hat{\jmath})$

Ex 3 vectors a, b, c are coplanar iff they span a parallelepiped of zero volume iff $\det([a\ b\ c]) = 0$.

The Cross Product in $\mathbb{R}^3$

Def (Geometric) Let $\vec{u}, \vec{v}$ be vectors in $\mathbb{R}^3$. The cross product $u \times v$ is the vector in $\mathbb{R}^3$ satisfying:

(1) perpendicular to both of u and v,
(2) $|u \times v| = |u| \cdot |v| \sin\theta$, θ = angle between u and v,
(3) direction determined by the right hand rule.



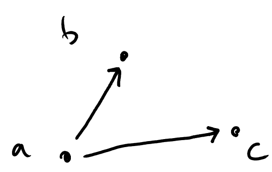
$|u \times v|$ = area of the parallelogram

★ Thm (Formula for cross product)

$$(a,b,c) \times (x,y,z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ x & y & z \end{vmatrix}$$

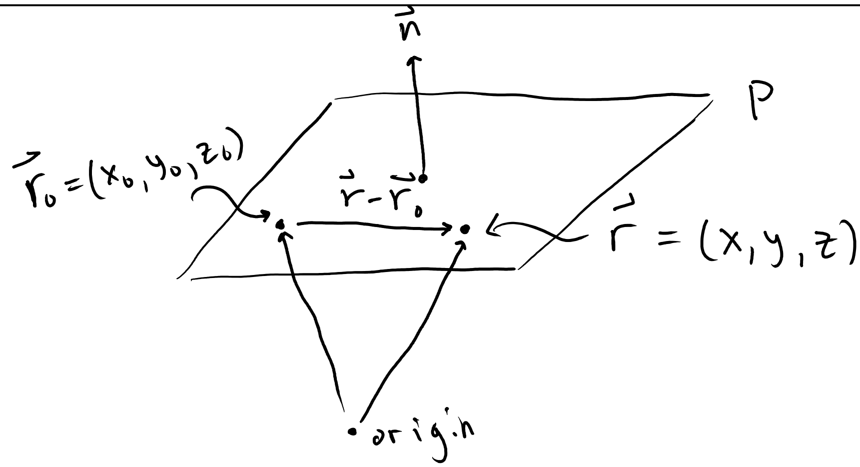
★ Thm Vectors u and v are parallel $\iff |u \times v| = 0$
$\iff u \times v = \vec{0}$

Application Three points $a, b, c \in \mathbb{R}^3$ are collinear iff $\overrightarrow{b-a}$ and $\overrightarrow{c-a}$ have cross-product $= 0$. (compare w/ ex from last week)



□

The Equation of a Plane



- Let $\vec{n}$ be any vector perpindicular to P.
- Let $r_0 = (x_0, y_0, z_0)$ be any point in P.

Assuming $r = (x, y, z)$ lies in P, then $\vec{r} - \vec{r}_0$ lies in P. Then $\vec{r} - \vec{r}_0$ is orthogonal to $\vec{n}$. Hence,

$$\boxed{\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0}$$

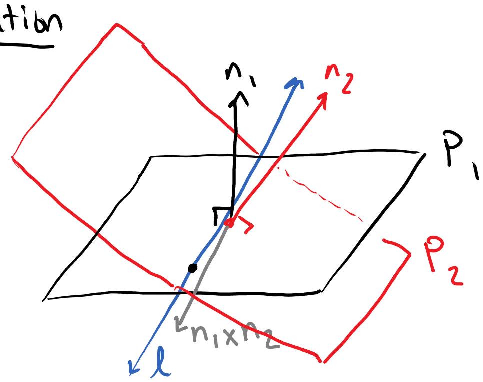
Expand this formula:

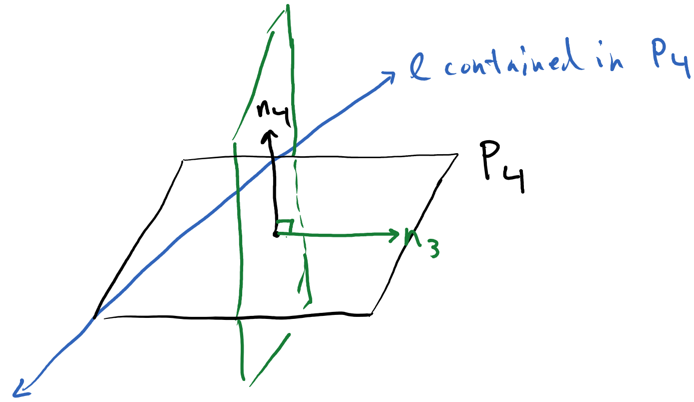
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Problem 1 | **Chapter 11.3**

Find an eq. of the plane that passes through the line of intersection of $P_1: x - z = 1$ and $P_2: y + 2z = 3$, and is perpendicular to $P_3: x + y - 2z = 1$.

Solution





Let P_4 be the plane we are looking for. P_4 contains ℓ so it must contain a point on ℓ. A point on ℓ lies in P_1, and P_2. We can take $r_0 = (1,3,0)$ as a point in P_4.

Next, we find a normal vector n_4 for P_4. Since $P_3 \perp P_4$, we must have $n_3 \perp n_4$. Since P_4 contains ℓ, it must be parallel to the direction of ℓ, $n_1 \times n_2$, or in other words $n_4 \perp n_1 \times n_2$.

So we have
$$n_4 = n_3 \times (n_1 \times n_2)$$
$$= (n_3 \cdot n_2)\, n_1 - (n_3 \cdot n_1)\, n_2 \qquad \text{(in the book)}$$

$$n_3 \cdot n_2 = (1,1,-2)\cdot(0,1,2) = -3$$

$$n_3 \cdot n_1 = (1,1,-2)\cdot(1,0,-1) = 3$$

$$= -3(1,0,-1) - 3(0,1,2)$$
$$= (-3,0,3) - (0,3,6)$$
$$= (-3,-3,-3).$$

$$r_0 = (1,3,0)$$

So an eq. of the plane is:

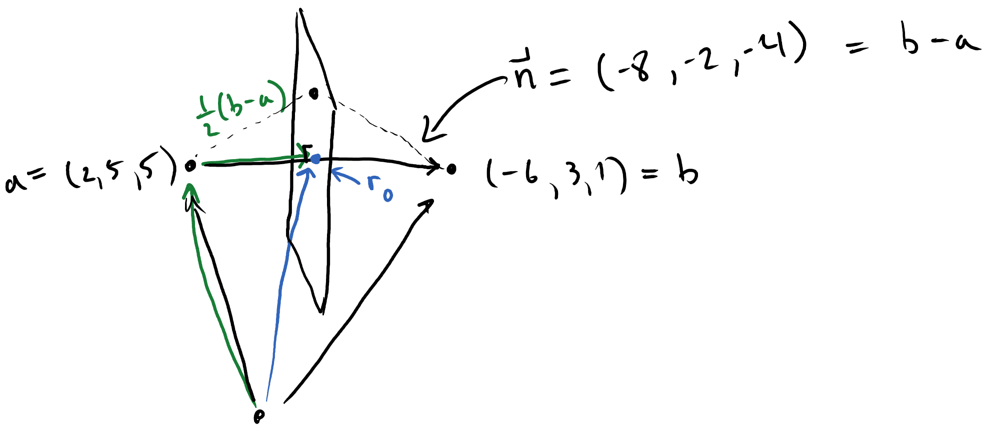
$$(-3,-3,-3)\cdot\big((x,y,z) - (1,3,0)\big) = 0$$

$$\Rightarrow\ -3(x-1) - 3(y-3) - 3(z) = 0 \ \Rightarrow\ \boxed{x + y + z = 4}$$

Problem 2 | **Chapter 11.3**

Find an equation of the plane that is equidistant from the points $(2,5,5)$ and $(-6,3,1)$.

<u>Solution</u>



So $r_0 = a + \frac{1}{2}(b-a) = \frac{1}{2}(b+a)$ (The midpoint of a & b)

$$= \frac{1}{2}(-4,8,6) = (-2,4,3)$$

So an equation of the plane is

$$\boxed{(-8,-2,-4)\cdot\big((x,y,z)-(-2,4,3)\big)=0}$$

$\blacksquare$

Problem 3

Chapter 11.3

(a) Show that the lines $r_1(t) = (1,1,0) + t(1,-1,2)$, $r_2(s) = (2,0,2) + s(-1,1,0)$ intersect.

(b) Find a plane containing both lines.