Final Exam Review

Friday, December 11, 2020 3:19 PM

1. (12 points) For each of the questions below, indicate if the statement is true (T) or false (F).

(a) Let $\mathbf{F}:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a vector field of class C^2. Then $\text{div}(\text{curl } \mathbf{F}) = \nabla \cdot (\nabla \times \mathbf{F}) = 0$.

Answer (T/F): T

(b) If f is a C^2 scalar function, then $\nabla \times (\nabla f) = \mathbf{0}$.

Answer (T/F): T

(c) Let $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function where all second order partial derivatives exist and are not continuous for all points $(x,y)\in\mathbb{R}^2$. Then $\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y)$ for all points $(x,y)\in\mathbb{R}^2$.

not C^2

Answer (T/F): F

(d) Let $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function where all second order partial derivatives exist and are continuous for all points $(x,y)\in\mathbb{R}^2$. Then $\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y)$ for all points $(x,y)\in\mathbb{R}^2$.

Clairaut's Theorem

Answer (T/F): T

(e) Let $f(x,y)$ be a C^2 function which has a local maximum at $(0,0)$. Then the Hessian matrix of f at $(0,0)$ is necessarily negative definite.

Answer (T/F): F

(f) Let $D\subset\mathbb{R}^2$ be a closed and bounded set. Every continuous function $f:D\longrightarrow\mathbb{R}$ has a global maximum and a global minimum value on D.

Answer (T/F): T

1. a) $$\nabla\cdot(\nabla\times F) = \nabla\cdot\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = \nabla\cdot\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 F_3}{\partial x\partial y}} - \cancel{\frac{\partial^2 F_2}{\partial x\partial z}} + \cancel{\frac{\partial^2 F_1}{\partial y\partial z}} - \cancel{\frac{\partial^2 F_3}{\partial y\partial x}} + \cancel{\frac{\partial^2 F_2}{\partial z\partial x}} - \cancel{\frac{\partial^2 F_1}{\partial z\partial y}} = 0$$

b) $f:\mathbb{R}^3\to\mathbb{R}$

$$\nabla\times\nabla f = \nabla\times\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = (0,0,0).$$

c) ✓ d) ✓

e) False. Consider $f(x,y) = -x^4$, $\frac{\partial^2 f}{\partial x^2}(x,y) = 12x^2$

so $\frac{\partial^2 f}{\partial x^2}(0,0) = 0$ which is not negative.

f) True.

2. (5 points) The equation of the tangent plane to the graph of $f(x,y) = 4 + x^2 + xy$ at the point $(1,1)$.

(A) $z = 3x+y$ (B) $z = 6+3(x-1)+(y-1)$ (C) $z = 3(x-1)+(y-1)$

(D) $z = 6+(2x+y)(x-1)+x(y-1)$ (E) $0 = 6+3(x-1)+(y-1)$ Answer (Letter): B

2) A normal vector is $\left(\frac{\partial}{\partial x}(1,1),\ \frac{\partial}{\partial y}(1,1),\ -1\right) = (3,1,-1)$

A point in the plane is $(1,1,f(1,1)) = (1,1,6)$

So an eq of the plane is

$$(3,1,-1)\cdot(x-1,\ y-1,\ z-6) = 0$$

$$\Rightarrow \boxed{3x-3+y-1-z+6=0}$$

Normal vector: The level set $f(x,y) - z = 0$ is the graph of f. Write $g(x,y,z) = f(x,y) - z$. By a Thm, ∇g is normal to the level set.

3. (5 points) Which of the following functions V can be considered a vector field. State all that apply.

(A) $V:\mathbb{R}^2\to\mathbb{R}^3$ (B) $V:\mathbb{R}^2\to\mathbb{R}^2$ (C) $V:\mathbb{R}^1\to\mathbb{R}^3$ (D) $V:\mathbb{R}^3\to\mathbb{R}^1$ (E) $V:\mathbb{R}^3\to\mathbb{R}^3$

Answer (Letter(s)): B, E

3) A vector field is a map of the form

$$f:\mathbb{R}^n\longrightarrow\mathbb{R}^n.$$

4. (6 points) Let S be the quadratic surface given by $4x^2+y^2+2z^2=10$.

(a) Classify S:

(A) S is an ellipsoid (B) S is a hyperboloid of one sheet (C) S is a hyperboloid of two sheets

(D) S is an elliptic cone (E) S is an elliptic paraboloid Answer (Letter): A

(b) Find a unit normal $\vec{N}$ to S at $(1,2,1)$

$\vec{N} = \left(\frac{2}{\sqrt6},\frac{1}{\sqrt6},\frac{1}{\sqrt6}\right)$

(c) Find the equation of the tangent plane to S at the point $(1,2,1)$

Answer: $2x+y+z=5$

4) b) Write $g(x,y,z) = 4x^2+y^2+2z^2$. Then

$$\nabla g(x,y,z) = (8x,\ 2y,\ 4z)$$

$$\Rightarrow \nabla g(1,2,1) = (8,4,4)$$

Use $N = \frac{1}{4}\nabla g(1,2,1) = (2,1,1)$. Normalize

$$N' = \frac{N}{\|N\|} = \frac{(2,1,1)}{\sqrt6} = \left(\frac{2}{\sqrt6},\frac{1}{\sqrt6},\frac{1}{\sqrt6}\right)$$

c) $$(2,1,1)\cdot(x-1,\ y-2,\ z-1) = 0$$

$$\Rightarrow 2x+y+z=5$$

5. (8 points) Consider the function

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq(0,0) \end{cases} \quad (1)$$

5) a) $$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to0}\frac{f(0+h,0)-f(0,0)}{h}$$

5. (8 points) Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \qquad (1)$$

(a) Compute $\frac{\partial f}{\partial x}(0,0)$ $\frac{\partial f}{\partial x}(0,0) =$ $\boxed{0}$

(b) Compute $\frac{\partial f}{\partial y}(0,0)$ $\frac{\partial f}{\partial y}(0,0) =$ $\boxed{0}$

For (c) and (d), state whether the statement is true (T) or false (F)

(c) The function $f(x,y)$ is continuous at $(0,0)$. Answer (T/F): $\boxed{T}$

(d) The function $f(x,y)$ is differentiable at $(0,0)$. Answer (T/F): $\boxed{}$

5) a) $\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$

$= \lim_{h\to 0} \frac{0 - 0}{h} = 0$

b) By symmetry, $\frac{\partial f}{\partial y}(0,0) = 0$

c) Check: $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \frac{r^2 \sin\theta\cos\theta}{r}$

$= \lim_{r\to 0} r\sin\theta\cos\theta = 0 = f(0,0).$

d) Method 1) use the definition:

$$\lim_{(x,y)\to(0,0)} \frac{\left| f(x,y) - \left(f(0,0) + Df(0,0)\begin{bmatrix} x-0 \\ y-0 \end{bmatrix}\right)\right|}{\|(x,y)\|}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{xy}{\sqrt{x^2+y^2}} - \left(0 + \left[\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)\right]\begin{bmatrix} x-0 \\ y-0 \end{bmatrix}\right)}{\sqrt{x^2+y^2}}$$

$= \lim_{(x,y)(0,0)} \frac{xy}{x^2+y^2}$, this limit does not exist.

Since the limit is not zero, f is not differentiable.

Method 2) Check that second partial derivatives are continuouse functions.

6. (4 points) Suppose $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a differentiable function which has an absolute maximum value M and an absolute minimum m. Suppose further that $m = M$. What can be said about its derivative $\mathbf{D}f$?

$\mathbf{D}f =$ $\boxed{[0,0,0]}$

6) By assumption, for all $(x,y,z) \in \mathbb{R}^3$

$M = m \leq f(x,y,z) \leq M$

This says $f(x,y,z) = M$. So $Df = [0,0,0]$.

7. (6 points) Find the (x,y,z) coordinates of the points P where the line $\mathbf{l}(t) = (x,y,z) = (1-t, 1+t, t)$ intersects the sphere $x^2+y^2+z^2 = 11$.

$P =$ $\boxed{l(\sqrt{3}),\ l(-\sqrt{3})}$

7) Solve for t:

$(1-t)^2 + (1+t)^2 + t^2 = 11$

$\Rightarrow 1 - \cancel{2t} + t^2 + 1 + \cancel{2t} + t^2 + t^2 = 11$

$\Rightarrow 3t^2 = 9 \Rightarrow t = \pm\sqrt{3}$

So $l(\sqrt{3}), l(-\sqrt{3})$ are the points where l intersects the sphere.

8. (6 points) Consider $f(x,y) = -y^2 + x^2y + xy$. The three crtitical points are

(i) (0,0) (ii) (-1,0) (iii) $(-\frac{1}{2}, -\frac{1}{8})$

State whether the critical points given in (i), (ii), and (iii) are a local maximum (MAX), local minimum (MIN) or saddle point (SAD).

(i) $\boxed{\text{SAD}}$ (ii) $\boxed{\text{SAD}}$ (iii) $\boxed{\text{Max}}$

8) Compute the Hessian.

$\frac{\partial f}{\partial x} = 2xy + y$ $\qquad \frac{\partial^2 f}{\partial x^2} = 2y$ $\qquad \frac{\partial^2 f}{\partial x \partial y} = 2x+1$

$\frac{\partial f}{\partial y} = -2y + x^2 + x$ $\qquad \frac{\partial^2 f}{\partial y^2} = -2$

$$\frac{\partial f}{\partial y} = -2y + x^2 + x \qquad \frac{\partial^2 f}{\partial y^2} = -2$$

So $$D(x,y) = \begin{vmatrix} 2y & 2x+1 \\ 2x+1 & -2 \end{vmatrix} = -4y - (2x+1)^2$$

i) $D(0,0) = -1 < 0$

ii) $D(-1,0) = -1 < 0$

iii) $D(-\tfrac{1}{2},-\tfrac{1}{8}) = -4(-\tfrac{1}{8}) - (2(-\tfrac{1}{2})+1)^2$
$= \tfrac{1}{2} > 0$

Also, $\frac{\partial^2 f}{\partial x^2}(-\tfrac{1}{2},-\tfrac{1}{8}) = 2(-\tfrac{1}{8}) = -\tfrac{1}{4} < 0$

So the Hessian is neg. definite.

9. (5 points) $f(x,y) = x^2 + y^2 - xy$ has a minimum at $(0,0)$. $g(x,y) = x^2 + y^2 - 3xy$ has a saddle point at $(0,0)$. For all $\lambda \in \mathbb{R}$, $h(x,y) = x^2 + y^2 - \lambda xy$ has a critical point at $(0,0)$. There is a number λ_0 such that for $\lambda < \lambda_0$, $(0,0)$ is a minimum and for $\lambda > \lambda_0$, $(0,0)$ is a saddle point. Find λ_0.

$\lambda_0 =$ ± 2

9. Use the Second derivative test:

$$\frac{\partial h}{\partial x} = 2x - \lambda y \qquad \frac{\partial^2 h}{\partial x^2} = 2 \qquad \frac{\partial^2 h}{\partial x \partial y} = -\lambda$$

$$\frac{\partial h}{\partial y} = 2y - \lambda x \qquad \frac{\partial^2 h}{\partial y^2} = 2$$

So then
$$D(0,0) = \begin{vmatrix} 2 & -\lambda \\ -\lambda & 2 \end{vmatrix} = 4 - \lambda^2$$

If $D(0,0) > 0$, then $(0,0)$ is a min.
If $D(0,0) < 0$, then $(0,0)$ is a saddle.

So we solve $0 = D(0,0) = 4 - \lambda^2 \Rightarrow \lambda = \pm 2$.

10. (5 points) Use Lagrange multipliers to find the maximum and minimum values of $f(x,y) = 2x + y$ on the unit circle $x^2 + y^2 = 1$. Minimum Maximum

10. Solve the system of equations:

$$(2,1) = \nabla f(x,y) = \lambda \nabla g(x,y) = \lambda(2x, 2y)$$

$$\begin{cases} 1 = \lambda x & (1) \\ 1 = 2\lambda y & (2) \\ x^2 + y^2 = 1 & (3) \end{cases}$$

By (1), $\lambda \neq 0$. Then using (1), (2), we have

$$x = \frac{1}{\lambda} = 2y. \quad (*)$$

Using $(*)$ with (3) : $4y^2 + y^2 = 1 \Rightarrow y = \pm \frac{1}{\sqrt{5}}$

Then by $(*)$, $x = \pm \frac{2}{\sqrt{5}}$. The max is then

$$f\left(\tfrac{2}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}\right) = \sqrt{5}, \quad f\left(-\tfrac{2}{\sqrt{5}}, -\tfrac{1}{\sqrt{5}}\right) = -\sqrt{5}$$

11. (5 points) Find the distance of the origin to the plane $\mathbb{P}$: $x + y + z = 1$ (meaning the shortest distance among all points P on $\mathbb{P}$ to $(0,0,0)$). You may use the formula or Lagrange multipliers.

Answer: $1/\sqrt{3}$

11. Maximize $d^2(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $x + y + z = 1$. By Lagrange,

11. (5 points) Find the distance of the origin to the plane $\mathbb{P}$: $x+y+z=1$ (meaning the shortest distance among all points P on $\mathbb{P}$ to $(0,0,0)$). You may use the formula or Lagrange multipliers.

Answer: $1/\sqrt{3}$

11. Maximize $d^2(x,y,z) = x^2+y^2+z^2$ subject to the constraint $x+y+z=1$. By Lagrange,

$$(2x, 2y, 2z) = \lambda(1,1,1)$$

so
$$\begin{cases} 2x = \lambda & (1) \\ 2y = \lambda & (2) \\ 2z = \lambda & (3) \\ x+y+z=1 & (4) \end{cases}$$

By (1), (2), (3), we get $x = y = z$

Hence, by (4),

$3x = 1$ so $x = \frac{1}{3} = y = z$.

So we have a single critical point $(1/3, 1/3, 1/3)$.
So the minimum distance is

$$\sqrt{d^2(1/3,1/3,1/3)} = \sqrt{1/3} = \frac{1}{\sqrt{3}}.$$

12. (9 points) Let $f(u,v) = (u-v, 1, u+v)$ and $g(x,y,z) = xyz$. Find $\mathbf{D}f(u,v)$, $\mathbf{D}g(x,y)$ and use the chain rule to compute the derivative $\mathbf{D}(g \circ f)(0,1)$.

$\mathbf{D}f(u,v) = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$ $\mathbf{D}g(x,y) = (yz \quad xz \quad xy)$ $\mathbf{D}(g\circ f)(0,1) = [0, -2]$

12.
$$Df(u,v) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$Dg(x,y,z) = \begin{bmatrix} yz & xz & xy \end{bmatrix}$$

Use Chain Rule:

$$\begin{aligned} D(g\circ f)(0,1) &= Dg(f(0,1)) \cdot Df(0,1) \\ &= Dg(-1,1,1) \cdot Df(0,1) \\ &= [1, -1, -1] \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = [0 \quad -2] \end{aligned}$$

13. (9 points) Let $f(x,y) = \cos(xy)$. Compute the following:

(a) $\nabla f(x,y)$

$\nabla f(x,y) =$

(b) The direction of the fastest rate of increase of f at the point $P(1, \frac{\pi}{2})$. Give your answer as a unit vector.

(c) The maximal value of the directional derivative at $P(1, \frac{\pi}{2})$.

13. a) $\nabla f(x,y) = (-y\sin(xy), -x\sin(xy))$.

b) $\nabla f(1, \pi/2) = \left(-\frac{\pi}{2}\sin\frac{\pi}{2}, -\sin \pi/2\right)$
$= \left(-\frac{\pi}{2}, 1\right)$.

Normalize: $\dfrac{(-\pi/2, 1)}{\sqrt{\pi^2/4 + 1}} = \left(\dfrac{-\pi}{2\sqrt{\pi^2/4+1}}, \dfrac{1}{\sqrt{\pi^2/4+1}}\right)$

c) $\|\nabla f(1, \pi/2)\| = \sqrt{\pi^2/4 + 1}$

14. (9 points) A vector field $\mathbf{F}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is defined by $\mathbf{F}(x,y,z) = (-x, y, 2)$. Compute the following:

(a) (3 points) $\mathrm{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} =$ 0

(b) (3 points) $\mathrm{curl}(\mathbf{F}) = \nabla \times \mathbf{F} =$ $(0,0,0)$

(c) (2 points) True or false: Is $\mathbf{F}$ a gradient vector field ?

14. a) $\nabla \cdot F = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \cdot (-x, y, 2)$

$$= \frac{\partial(-x)}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial}{\partial z} 2$$

(b) (3 points) $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} =$ $\boxed{(0,0,0)}$

(c) (2 points) True or false: Is $\mathbf{F}$ a gradient vector field ? $\boxed{}$

$$= \frac{\partial(-x)}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial}{\partial z} z$$

$$= -1 + 1 + 0 = 0$$

b) $$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -x & y & z \end{vmatrix}$$

$$= (0-0,\ 0-0,\ 0-0) = (0,0,0)$$

c) A gradient vector field is of the form

$F = \nabla f$ for some C^2 fnc $f : \mathbb{R}^3 \to \mathbb{R}$.

So F is a gradient vector field since

$$\nabla \times F = 0$$

$\blacksquare$

15. (8 points)

(a) Find the first order Taylor polynomial (or Taylor approximation) of $f(x,y) = xe^y$ at $(0,0)$.

$T_1(x,y) =$ $\boxed{x}$

(b) Find the second order Taylor polynomial (or Taylor approximation) of $f(x,y) = xe^y$ at $(0,0)$.

$T_2(x,y) =$ $\boxed{x + xy}$

15. $$\frac{\partial f}{\partial x} = e^y \longrightarrow \frac{\partial f}{\partial x}(0,0) = 1$$

$$\frac{\partial f}{\partial y} = xe^y \longrightarrow \frac{\partial f}{\partial y}(0,0) = 0$$

$$\frac{\partial^2 f}{\partial x^2} = 0 \longrightarrow \frac{\partial^2 f}{\partial x^2}(0,0) = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^y \longrightarrow \frac{\partial^2 f}{\partial x \partial y}(0,0) = 1$$

$$\frac{\partial^2 f}{\partial y^2} = xe^y \longrightarrow \frac{\partial^2 f}{\partial y^2}(0,0) = 0$$

Then $$T_2 f(0,0)(x,y) = \underbrace{f(0,0) + f_x(0,0)\,x + f_y(0,0)\,y}_{T_1 f(0,0)(x,y)} + \frac{1}{2}\left(f_{xx}(0,0)\,x^2 + 2 f_{xy}(0,0)\,xy + f_{yy}(0,0)\,y^2\right)$$

$$= 0 + x + 0 + \frac{1}{2}(0 + 2xy + 0)$$

$$= x + xy.$$

$\blacksquare$