Braid graphs in simply-laced triangle-free Coxeter systems are partial cubes

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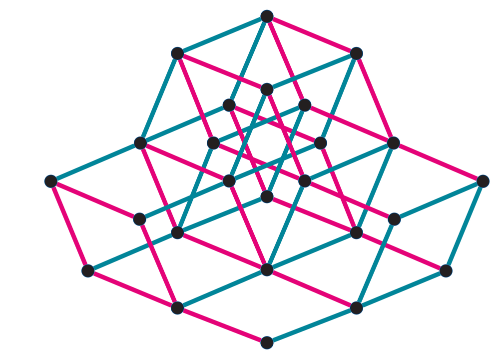


Simply-Laced Coxeter systems

A **simply-laced Coxeter system** is a pair (W, S) where S is a finite set and W is a group with presentation

$$W = \langle S \mid (st)^{m(s,t)} = 1 \rangle$$

where $m(s,t) \in \{1,2,3\}$ and $m(s,t) = 1$ iff $s = t$.

- $m(s,t) = 2 \implies st = ts$ (commutation relation)
- $m(s,t) = 3 \implies sts = tst$ (braid relation)

Coxeter graphs

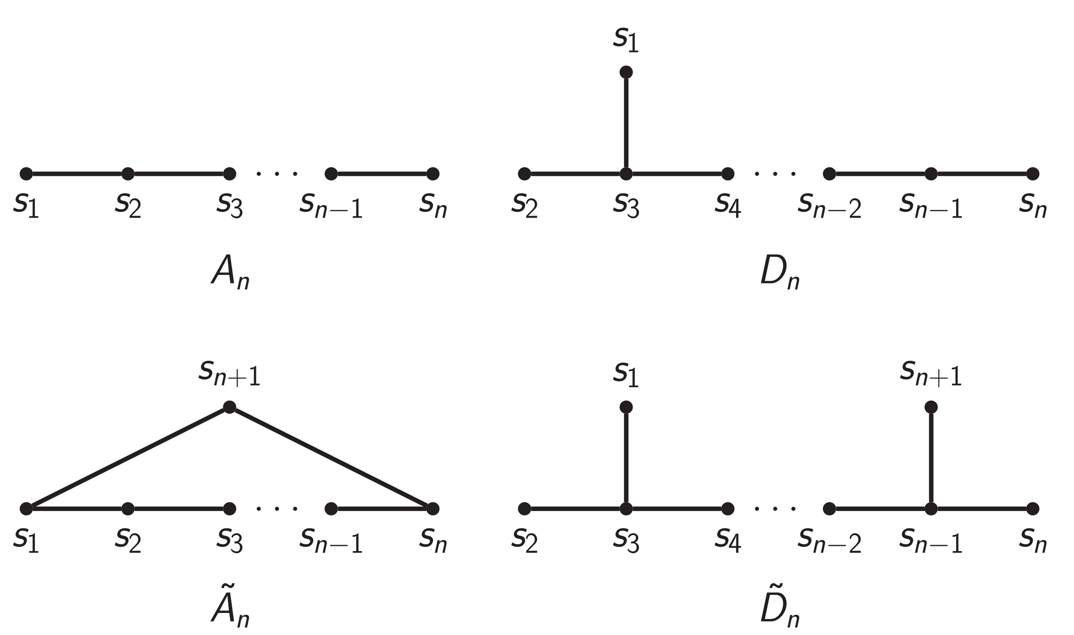
Every simply-laced Coxeter system (W, S) can be encoded by a unique **Coxeter graph** $\Gamma = \Gamma(W, S)$ with

- vertices S
- edges $\{s,t\}$ iff $m(s,t) = 3$.

We say that (W, S) is **triangle free** if the Coxeter graph Γ has no cycles of length three.

Examples of simply-laced Coxeter systems

Here are some examples of common (families of) simply-laced Coxeter systems. All of these examples are triangle-free, except for $\tilde{A}_2$.



Reduced Expressions

Let (W, S) be a Coxeter system and S^* the free monoid on $S = \{s_1, \ldots, s_n\}$ and let $\pi : S^* \to W$ be the natural monoid morphism.

- An **expression** for $w \in S$ is a word $\alpha = s_{x_1} \cdots s_{x_m} \in S^*$ satisfying $\pi(\alpha) = w$.
- If m is minimal among all expressions for w, then α is called a **reduced expression** for w. The set of reduced expressions for w is denoted by $\mathcal{R}(w)$.

When working with examples, we write $x_1 \cdots x_m$ instead of $s_{x_1} \cdots s_{x_m}$ using the bijection $S \to \{1, \ldots, n\}$, $s_i \mapsto i$.

Theorem (Matsumoto)

Any two reduced expressions for $w \in W$ are related by a finite sequence of commutation and braid relations.

Matsumoto graph

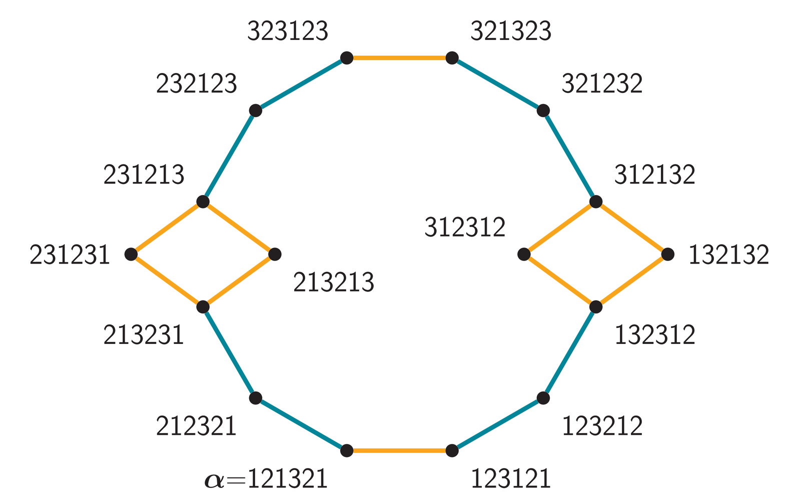
The **Matsumoto graph** for $w \in W$ is the graph $\mathcal{M}(w)$ with

- vertices $\mathcal{R}(w)$
- edges $\{\alpha, \beta\}$ iff α and β differ by a single commutation or braid relation.

Matsumoto's theorem implies that $\mathcal{M}(w)$ is a finite connected graph. It is well known that $\mathcal{M}(w)$ is bipartite.

Example of a Matsumoto graph

Consider the reduced expression $\alpha = 121321$ for $w \in W(A_3)$. Then $\mathcal{M}(w)$ is as follows:



Braid graphs

We say $\alpha, \beta \in \mathcal{R}(w)$ are **braid equivalent** if they are related by a sequence of braid relations. A braid equivalence class is denoted by $[\alpha]$. The **braid graph** for α is the graph $\mathcal{B}(\alpha)$ with

- vertices $[\alpha]$
- edges $\{\beta, \gamma\}$ iff β and γ are related by a single braid relation.

Braid graphs are the maximal blue connected subgraphs of $\mathcal{M}(w)$.

Braid shadows and Links

Let $\alpha = s_{x_1} \cdots s_{x_m}$ be a reduced expression in a simply-laced Coxeter system. Notation: $[\![i,k]\!] := \{i, i+1, \ldots, k-1, k\}$ for $1 \le i \le k$.

- Expressions of the form $\alpha_{[\![i,k]\!]} := s_{x_i} \cdots s_{x_k}$ are called **factors** of α. The factors of α are partially ordered in the obvious way.
- $[\![i, i+2]\!]$ is a **braid shadow** for α if $\alpha_{[\![i,i+2]\!]} = sts$ with $m(s,t) = 3$. The set of braid shadows for α is $\mathcal{S}(\alpha)$ and we set $S([\alpha]) = \cup_{\beta \in [\alpha]} \mathcal{S}(\beta)$.
- α is a **link** of rank $r \ge 0$ if $m = 2r + 1$ is odd and
$$\mathcal{S}([\alpha]) = \{[\![2i-1, 2i+1]\!] : 1 \le i \le r\}.$$
If α is a link, then $[\alpha]$ is called a **braid chain**.
- A factor β of α is a **link factor** if it is a link which is maximal among all factors of α with that property.

Theorem (Awik, B., Cadman, Ernst)

*Every nonidentity reduced expression α in a simply-laced Coxeter system can be written uniquely as a product $\alpha = \alpha_1 \cdots \alpha_k$ of link factors. This is the **link factorization** for α, which is denoted by $\alpha = \alpha_1 \mid \cdots \mid \alpha_k$.*

Example of links and link factors

Consider the reduced expression $\alpha_1 = 12132454$ in the Coxeter system of type A_5. The link factorizations of the elements of $[\alpha_1]$ are shown below.

$$\alpha_1 = 12132 \mid 454, \ \alpha_2 = 21232 \mid 454, \ \alpha_3 = 21323 \mid 454,$$

$$\alpha_4 = 12132 \mid 545, \ \alpha_5 = 21232 \mid 545, \ \alpha_6 = 21323 \mid 545.$$

The braid shadows are indicated by horizontal lines.

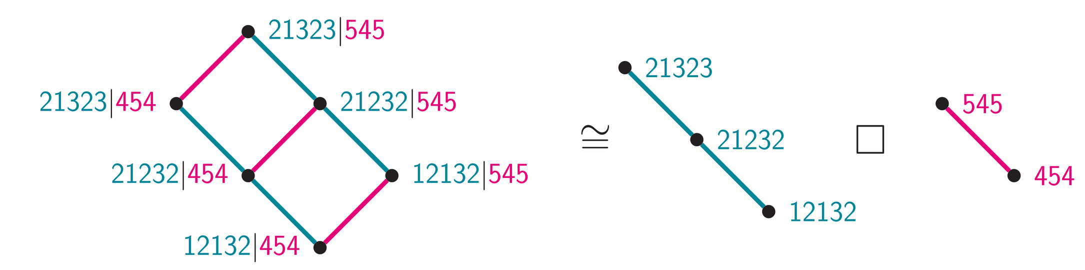
Theorem (Awik, B., Cadman, Ernst)

Let $\alpha = \alpha_1 \mid \cdots \mid \alpha_k$ be a reduced expression in a simply-laced Coxeter system. Then $[\alpha] = \{\beta_1 \mid \cdots \mid \beta_k : \beta_i \in [\alpha_i], \ 1 \le i \le k\}$. Moreover, there is a graph isomorphism

$$\mathcal{B}(\alpha) \cong \mathcal{B}(\alpha_1) \,\square \cdots \square\, \mathcal{B}(\alpha_k).$$

Example of a braid graph decomposition

The braid graph decomposition $\mathcal{B}(12132 \mid 454) \cong \mathcal{B}(12132) \,\square\, \mathcal{B}(454)$ is shown below:



Signature of a link

Let $\alpha = s_{x_1} \cdots s_{x_{2r+1}}$ be a link of rank $r \ge 1$. The **signature** of α is the finite sequence $\mathrm{sig}(\alpha) = (s_{x_2}, \ldots, s_{x_{2r}})$ of generators appearing at the even indices of α. The ith entry of $\mathrm{sig}(\alpha)$ is denoted by $\mathrm{sig}_i(\alpha)$.

Theorem (Awik, B., Cadman, Ernst)

Let α be a link of rank $r \ge 1$ in a simply-laced triangle-free Coxeter system. Then

(a) *For all $1 \le i \le r$, there exists a unique pair $\{s_i, t_i\} \subseteq S$ satisfying $m(s_i, t_i) = 3$ and $\mathrm{sig}_i(\beta) \in \{s_i, t_i\}$ for all $\beta \in [\alpha]$.*

(b) *Suppose $\beta \in [\alpha]$. Then $\mathrm{sig}(\alpha) = \mathrm{sig}(\beta)$ if and only if $\alpha = \beta$.*

Hypercube graph

Let $r \ge 1$. The **hypercube** graph of dimension r is the graph Q_r with

- vertices $\{0,1\}^r = \{\mathbf{a} := (a_1, \ldots, a_r) : a_i \in \{0,1\}\}$ (binary strings of length r)
- edges $\{\mathbf{a}, \mathbf{b}\}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ differ by a single digit.

The **Hamming distance** makes Q_r into a metric space.

Partial cubes

Let $G = (V, E)$ be a finite connected graph. Then G is a metric space with metric $d_G(u,v)$ defined to be the length of any shortest path between vertices $u, v \in V$

- We call G a **partial cube** if there is an isometric embedding $G \to Q_r$ for some $r \ge 0$.
- The box product of two partial cubes is a partial cube.

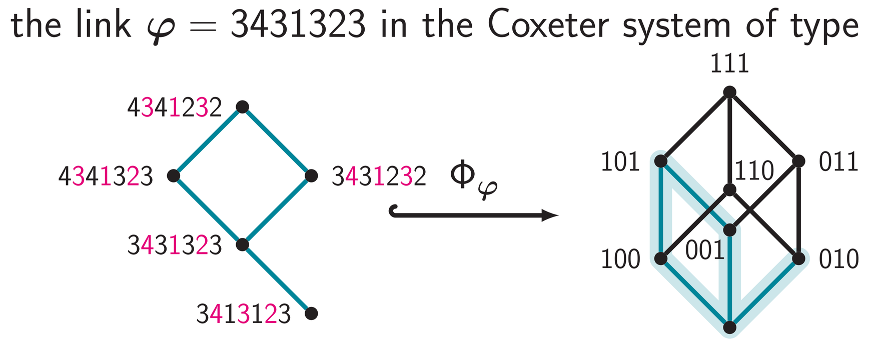
An isometric embedding for a link

Let α be a link of rank $r \ge 1$. For each $1 \le i \le j$, let $\{s_i, t_i\} \subset S$ be as in (a) of the preceding theorem. Define a map $\Phi_\alpha : [\alpha] \to \{0,1\}^r$ via $\Phi_\alpha(\beta) = a_1 \cdots a_r$ where

$$a_i = \begin{cases} 0, & \mathrm{sig}_i(\beta) = \mathrm{sig}_i(\alpha) \\ 1, & \mathrm{sig}_i(\beta) \neq \mathrm{sig}_i(\alpha). \end{cases}$$

Example of the embedding

Consider the link $\varphi = 3431323$ in the Coxeter system of type D_4.



The isometric subgraph on the right is actually a well-known graph, known as a *Fibonacci cube*. See the definition below.

Theorem (Awik, B., Cadman, Ernst)

Suppose that α is a link of rank $r \ge 1$ in a simply-laced triangle-free Coxeter system. Then Φ_α is an isometric embedding of $\mathcal{B}(\alpha)$ into Q_r. Thus, braid graphs in simply-laced triangle-free Coxeter systems are partial cubes.

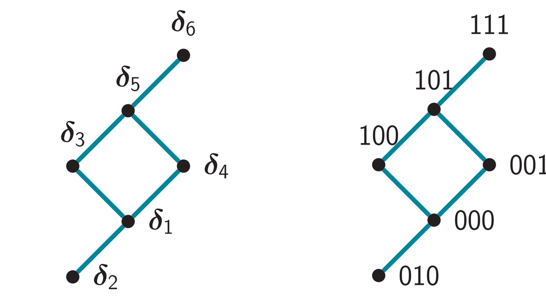
Why triangle-free?

Consider reduced expression $\delta_1 = 1213121$ in type $\widetilde{A}_2$ with braid class:

$$\delta_1 = 1213121, \ \delta_2 = 1231321$$

$$\delta_3 = 2123121 \quad \delta_4 = 1213212$$

$$\delta_5 = 2123212 \quad \delta_6 = 2132312$$



Notice: The map Φ_{δ_1} is not an isometry since the Hamming distance of 010 and 111 is two. The problem is that $\mathrm{sig}_2(\delta_i)$ has three possible values! However, $\mathcal{B}(\delta_1)$ is a partial cube - it can be isometrically embedded into Q_4.

Fibonacci cubes

The **Fibonacci cube** of rank $r \ge 1$ is the subgraph of $\mathcal{F}_r$ of Q_r induced by the set of vertices

$$V_r = \{a_1 \cdots a_r \in \{0,1\}^r : a_i a_{i+1} = 0, \ 1 \le i \le r-1\}.$$

The Fibonacci cube $\mathcal{F}_r$ is a partial cube having F_{r+2} many vertices, where F_n is the n-th Fibonacci number.

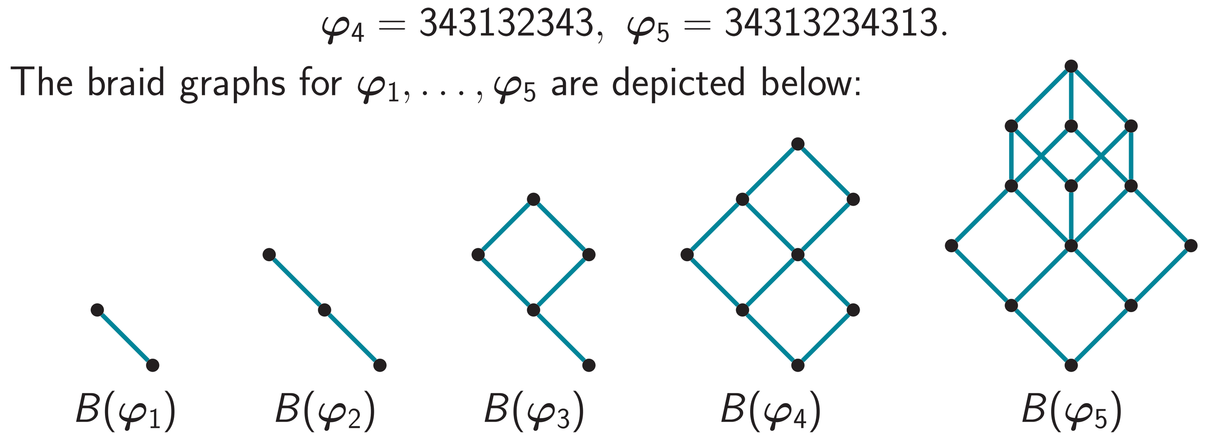
Fibonacci links

A link φ of rank $r \ge 1$ is called a **Fibonacci** link if $|\mathcal{S}(\varphi)| = r$. The following are Fibonacci links of ranks $1-5$ in the Coxeter system of type D_4:

$$\varphi_1 = 343, \ \varphi_2 = 34313, \ \varphi_3 = 3431323,$$

$$\varphi_4 = 343132343, \ \varphi_5 = 34313234313.$$

The braid graphs for $\varphi_1, \ldots, \varphi_5$ are depicted below:

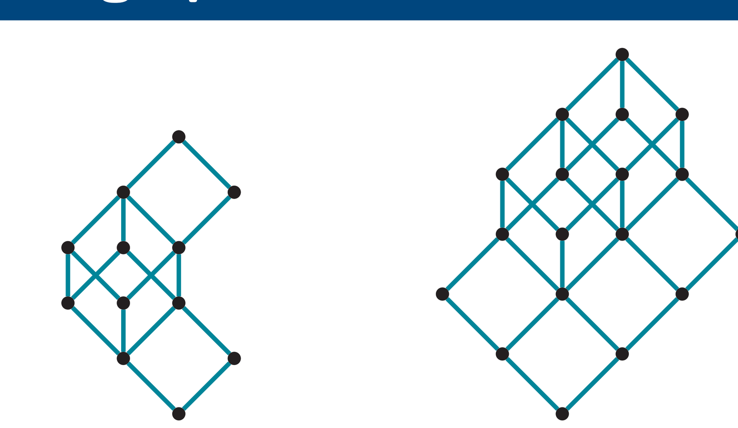


These are Fibonacci cubes! This is not a coincidence.

Theorem (Awik, B., Cadman, Ernst)

Let φ be a Fibonacci link of rank $r \ge 1$ in a simply-laced triangle-free Coxeter system. Then $\mathcal{B}(\varphi)$ is isomorphic to the Fibonacci cube $\mathcal{F}_r$.

More braid graphs



References

F. Awik, J. Breland et al. Braid graphs in simply-laced triangle-free Coxeter systems are partial cubes. *European Journal of Combinatorics* 118, 2024.

Based on joint work with F. Awik, Q, Cadman, and D.C. Ernst