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Taxonomy of Braid Graphs

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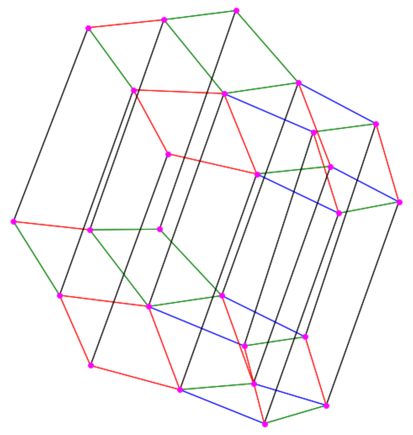
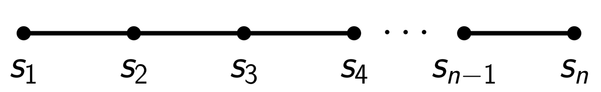


Coxeter Systems of Type A

A Coxeter system of type A_n ($n \geq 1$) is a pair consisting of a Coxeter group $W(A_n)$ and a finite set $S = \{s_1, s_2, \dots, s_n\}$ of generating involutions subject to the relations:

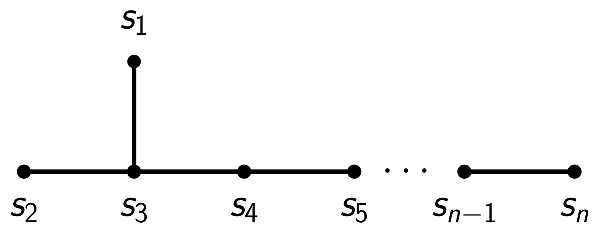
- $s_is_i = e$ for all i;
- $s_is_j = s_js_i$ when $|i-j| > 1$; (commutation relation)
- $s_is_js_i = s_js_is_j$ when $|i-j| = 1$. (braid relation)

The relations can be encoded into a graph called a Coxeter graph.



Coxeter Systems of Type D

The Coxeter system of type D_n has defining relations determined by the following Coxeter graph.



The Coxeter group $W(D_n)$ can be thought of as the group that acts on n coins by rearranging and flipping over evenly many.

Reduced Expressions

If a word $\overline{w} = s_{x_1}s_{x_2}\cdots s_{x_m} \in S^*$ is equal to w when considered as an element of W, we say that $\overline{w}$ is an expression for $w \in W$. If m is minimal among all expressions for w, then $\overline{w}$ is called a reduced expression.

Example 1

Consider the expression $\overline{w} = 4234354$ for $w \in W(A_5)$. We see that

$$42\mathbf{343}54 = 4\mathbf{242}3454 = 2\mathbf{44}3454 = 23454.$$

Then $\overline{w}$ is not reduced, but the expression on the right is.

Matsumoto's Theorem

Any two reduced expressions for w differ by a sequence of commutation and braid moves.

Braid Equivalence

If $\overline{w}_1$ and $\overline{w}_2$ are reduced expressions for $w \in W$, then $\overline{w}_1$ and $\overline{w}_2$ are braid equivalent if and only if we can obtain $\overline{w}_2$ from $\overline{w}_1$ via a sequence of braid moves. Equivalence classes are braid classes, denoted $[\overline{w}]$.

Example 2

Consider the reduced expressions $\overline{w}_1 = 3134323$ and $\overline{w}_2 = 1314232$ for $w \in W(D_4)$:

$$\underbrace{3134323}_{\overline{w}_1} = 1314323 = \underbrace{1314232}_{\overline{w}_2}$$

Applying all possible braid moves yields the braid class:

$$[\overline{w}_1] = \{1314232, 3134232, 3134323, 1314323, 3143423\}$$

Braid Links

Loosely speaking, a braid link is a reduced expression with the property that we can "slide" a braid across the whole expression.

Example 3

Consider the reduced expression below for $w \in W(A_6)$:

3435465 4**345**465 43**545**65 4354**656** 435465**767**

Then each reduced expression above is a braid link. The corresponding braid class is called a braid chain.

Example 4

Let $\overline{w} = 12132676$ be a reduced expression for $w \in W(A_7)$. Then $\overline{w}$ is not a link since we cannot slide a braid across the entire expression.

$$12132 \mid 676$$

However, we can see that $\overline{w}$ is a product of braid links.

Braid Link Factorizations

For a reduced expression $\overline{w}$, a consecutive subword is called a maximal braid link if it is a braid link that is not contained (with respect to position) in any other braid link in that word. A factorization into maximal braid links is called a braid link factorization.

Example 5

Consider reduced expressions $\overline{w}_1 = 12143465676$ and $\overline{w}_2 = 21234365767$ for some $w \in W(A_9)$:

$$\overline{w}_1 = 121 \mid 434 \mid 65676$$
$$\updownarrow \quad \updownarrow \quad \updownarrow$$
$$\overline{w}_2 = 212 \mid 343 \mid 65767$$

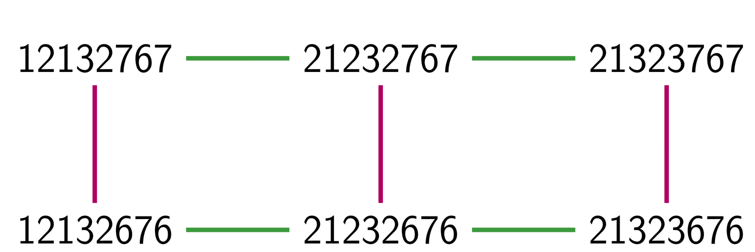
Braid Graphs

For a reduced expression $\overline{w}$ of $w \in W(A_n)$ or $w \in W(D_n)$ define the braid graph $B(\overline{w})$ via:

- Vertices: reduced expressions in $[\overline{w}]$.
- Edges: $\overline{w}_1$ and $\overline{w}_2$ are connected by an edge if and only if $\overline{w}_1$ and $\overline{w}_2$ are related via a single braid move.

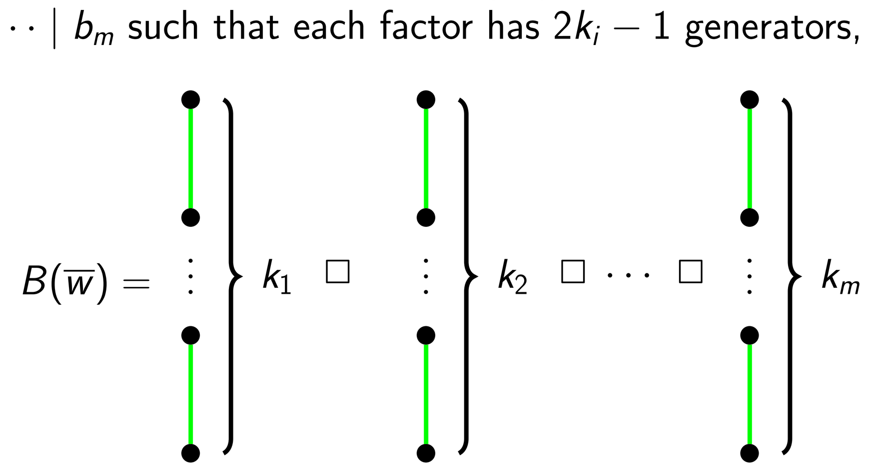
Example 6

Consider $\overline{w}$ from Example 4. Then $B(\overline{w})$ is as follows:



Theorem for Braid Graphs in Type A

If $\overline{w}$ is a reduced expression for $w \in W(A_n)$ having braid link factorization $b_1 \mid b_2 \mid \cdots \mid b_m$ such that each factor has $2k_i - 1$ generators, then

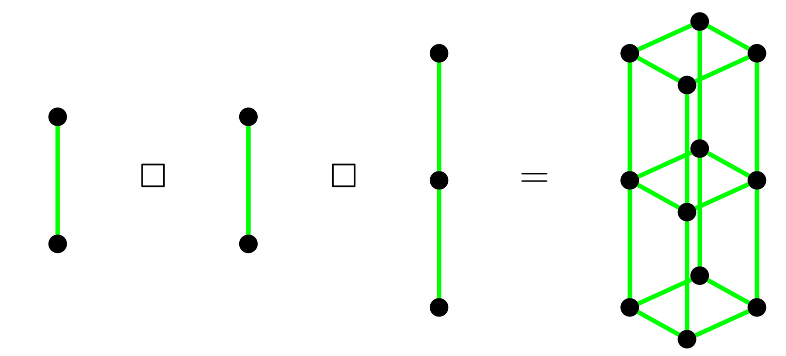


Example 7

Consider the reduced expression $\overline{w} = 12143465676$ for $w \in W(A_7)$, which has braid link factorization

$$121 \mid 434 \mid 65676.$$

The resulting braid graph for $\overline{w}$ is shown below:



Building Blocks in Braid Graphs of Type D

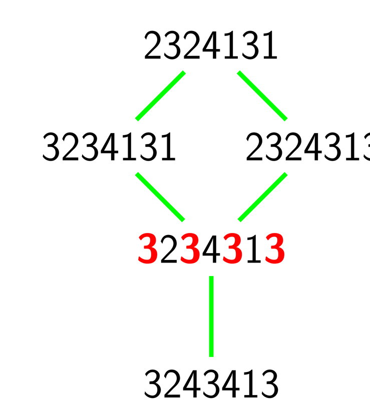
Let $\{a, b, c\} = \{1, 2, 4\}$. A braid graph $B(\overline{w})$ has the following forms:

- $B(\overline{w})$ has the form of a lollipop if there exists $3a3b3c3 \in [\overline{w}]$.
- $B(\overline{w})$ has the form of a glom if there exists $3a3b3c3a3 \in [\overline{w}]$.
- $B(\overline{w})$ has the form of a flapper box if there exists $3c3a3a3b3c3a3 \in [\overline{w}]$.

We believe that lollipops, gloms, flapper boxes, and paths are the "minimal building blocks" of braid graphs in Coxeter systems of type D.

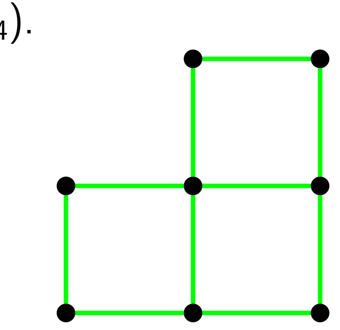
Lollipop

Below is a figure of the braid graph for 3234313, a reduced expression for $w \in W(D_4)$. Each vertex corresponds to a link, while the entire graph is a braid chain.



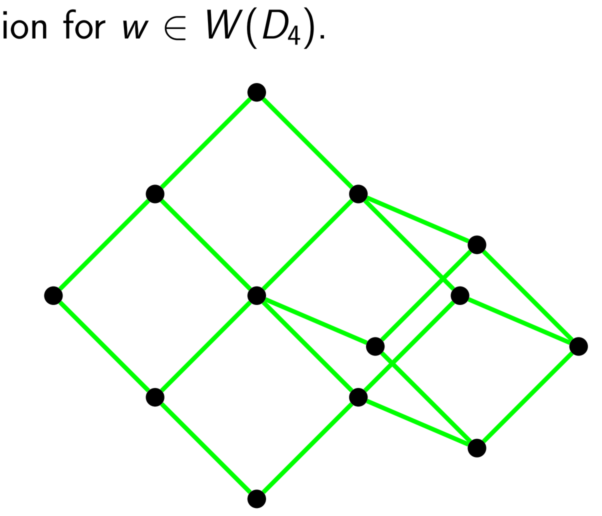
Glom

Below is an example of a glom which corresponds to 313234313, a reduced expression for $w \in W(D_4)$.



Flapper Box

Below is an example of a flapper box which corresponds to 31323431323, a reduced expression for $w \in W(D_4)$.

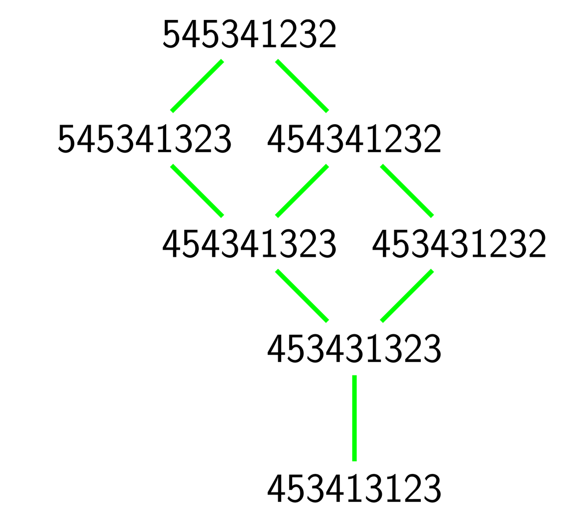


Type A Extensions

It turns out that we can extend lollipops, gloms, and flapper boxes on the left or on the right with appropriate links in type A.

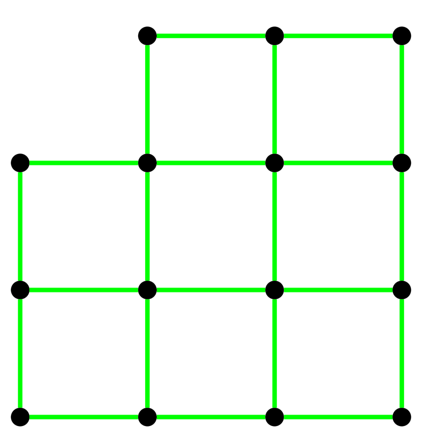
Extension of a Lollipop

Below is an example of a type A extension of a lollipop corresponding to 453431323, a reduced expression for $w \in W(D_5)$.



Extension of a Glom

Below is an example of a type A extension of a glom corresponding to 4534313234354, a reduced expression for $w \in W(D_5)$.



Conjecture for Braid Graphs in Type D

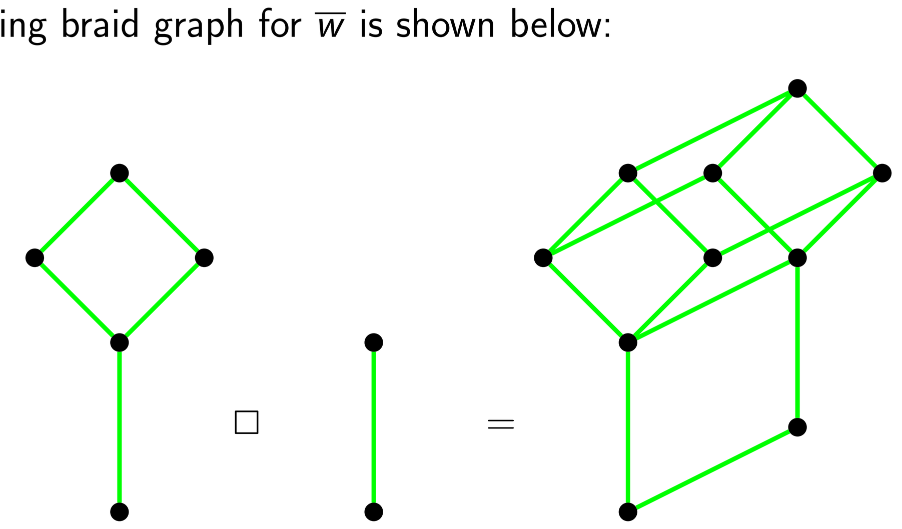
In type D, braid graphs are products of type A extensions of paths, lollipops, gloms, and flapper boxes corresponding to braid link factorizations.

Example 8

Consider the reduced expression $\overline{w} = 3234313767$ for $w \in W(D_7)$, which has a braid link factorization

$$3234313 \mid 767$$

The resulting braid graph for $\overline{w}$ is shown below:



Future Work

- Prove our conjecture about braid graphs in type D.
- Study the architecture of braid graphs in types A and D.
- Generalize to other Coxeter systems.

Directed by Dana C. Ernst