Architecture of braid classes in simply-laced Coxeter groups

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Joint work between Fadi Awik, **Jadyn Breland**, Quentin Cadman, and Dana Ernst **(ABCE)**

Northern Arizona University

Simply-laced Coxeter systems

Definition

A simply-laced Coxeter system is a pair (W, S) consisting of a finite set S of generators and a group W, called a Coxeter group, with presentation

$$W = \langle S \mid s^2 = e, (st)^{m(s,t)} = e \rangle,$$

with $m(s,t) \in \{2, 3\}$ for $s \neq t$.

Remark

The relations can be rewritten as follows.

$$\left.\begin{array}{l} m(s,t) = 2 \implies (st)^2 = e \implies st = ts \end{array}\right\} \quad \textit{commutation relation}$$

$$\left.\begin{array}{l} m(s,t) = 3 \implies (st)^3 = e \implies sts = tst \end{array}\right\} \quad \textit{braid relation}$$

Coxeter graphs

Definition

Let (W, S) be a simply-laced Coxeter system. A Coxeter graph is a graph Γ with

1. vertex set S, and;
2. edges $\{s, t\}$ if and only if $m(s, t) = 3$.

Remark

Equivalently, $\{s, t\}$ is an edge if and only if s, t satisfy the braid relation $sts = tst$.

Reduced Expressions

Definition

A word $\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_n} \in S^*$ is called an expression for w if it is equal to w when considered as an element of W. If n is minimal among all possible expressions for w, we say that $\overline{w}$ is a reduced expression for w.

Remark

1. *Each element $w \in W$ may have multiple reduced expressions that represent it.*
2. *The set of reduced expressions for $w \in W$ is denoted by $\mathcal{R}(w)$.*

Matsumoto's Theorem for simply-laced Coxeter systems

Definition

Let (W, S) be a simply-laced Coxeter system and $s, t \in S$.

1. If $m(s,t) = 2$, then $st = ts$. The replacement $st \mapsto ts$ is called a commutation.
2. If $m(s,t) = 3$, then $sts = tst$. The replacement $sts \mapsto tst$ is called a *braid move*.

Theorem (Matsumoto)

Let (W, S) be a simply-laced Coxeter system. Any two reduced expressions for the same group element $w \in W$ are related by a sequence of commutations and braid moves.

Reducing an expression

Example

The simply-laced Coxeter system of type A_3 is determined by the following graph.

Consider the expression 13212. This expression is not reduced. Let's reduce it.

$$13\color{blue}{212} = \color{blue}{13}121 = 3\color{blue}{11}21 = 321$$

The resulting expression 321 is reduced.

Braid equivalence

Definition

Let $\overline{w}_1, \overline{w}_2 \in \mathcal{R}(w)$. We say that $\overline{w}_1$ and $\overline{w}_2$ are braid equivalent if we can obtain $\overline{w}_2$ from $\overline{w}_1$ via a sequence of braid moves.

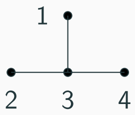
Remark

Braid equivalence is an equivalence relation on $\mathcal{R}(w)$. The corresponding equivalence classes are called braid classes and are denoted by $[\overline{w}]$.

Braid equivalent reduced expressions

Example

The simply-laced Coxeter system of type D_4 is determined by the following graph.



Consider the reduced expressions $\overline{w}_1 = 3134323$ and $\overline{w}_2 = 1314232$. **Claim:** $\overline{w}_1$ and $\overline{w}_2$ are braid equivalent.

$$\underbrace{3134323}_{\overline{w}_1} = 1314323 = \underbrace{1314232}_{\overline{w}_2}$$

Applying all possible braid moves yields the braid class:

$$[\overline{w}_1] = \{1314232, 3134232, 3134323, 1314323, 3143423\}$$

Braid shadows

Definition

Let $\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_n}$ be a reduced expression for $w \in W$. Then the interval $[i, i+2] \subset \mathbb{N}$ is a braid shadow if and only if $s_{x_i} = s_{x_{i+2}}$ and $m(s_{x_i}, s_{x_{i+1}}) = 3$.

Remark

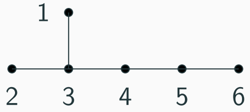
1. *The set of braid shadows for a particular reduced expression $\overline{w}$ is denoted $\mathcal{S}(\overline{w})$.*
2. *The set of braid shadows for the entire braid class $[\overline{w}]$ is the set*

$$\mathcal{S}([\overline{w}]) := \bigcup_{[\overline{w}]} \mathcal{S}(\overline{w}).$$

Braid shadows

Example

The simply-laced Coxeter system of type D_6 is determined by the following graph.



Consider the reduced expression $\overline{w} = 1314232565$ for some $w \in W(D_6)$.

$$\mathcal{S}(\overline{w}) = \{[1,3],[5,7],[8,10]\}$$

The set of braid shadows for $[\overline{w}]$ is the set: $\mathcal{S}([\overline{w}]) = \{[1,3],[3,5],[5,7],[8,10]\}$.

Braid chains

Definition

Let $\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_n}$ be a reduced expression for $w \in W$. We say that $\overline{w}$ is a link if and only if the following hold:

1. $[1, 3]$ and $[n-2, n]$ are braid shadows that each intersect another braid shadow in $\mathcal{S}([\overline{w}])$ and;
2. all other braid shadows in $\mathcal{S}([\overline{w}])$ intersect two other braid shadows in $\mathcal{S}([\overline{w}])$.

We say that the braid class $[\overline{w}]$ is a braid chain if and only if $\overline{w}$ is a link.

Example

The word $\overline{w} = 3134323$ is a reduced expression for some $w \in W(D_4)$. Then we have

$$\mathcal{S}([\overline{w}]) = \{[1, 3], [3, 5], [5, 7]\}.$$

According to the definition, $\overline{w}$ is a link and $[\overline{w}]$ is a braid chain.

Support

Definition

Let $\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_n}$ be a reduced expression for $w \in W$. We define the support as follows.

1. The support of $\overline{w}$ on the interval $[i,j] \subseteq \mathbb{N}$ is the set $\operatorname{supp}_{[i,j]}(\overline{w}) := \{s_{x_k} : i \leq k \leq j\}$.
2. The support of the braid class $[\overline{w}]$ on the interval $[i,j] \subseteq \mathbb{N}$ is the set

$$\operatorname{supp}_{[i,j]}([\overline{w}]) := \bigcup_{[\overline{w}]} \operatorname{supp}_{[i,j]}(\overline{w}).$$

Remark

The degenerate interval $[i,i] = \{i\}$ will be written i for simplicity.

Structure of $\mathcal{S}([\overline{w}])$

Theorem (ABCE)

Assume that (W,S) is a simply-laced Coxeter system. Let $\overline{w} = s_{x_1}s_{x_2}\cdots s_{x_n}$ be a reduced expression for some $w \in W$. If $[i,i+2] \in \mathcal{S}([\overline{w}])$, then $[i+1,i+3] \notin \mathcal{S}([\overline{w}])$.

Characterization of braid chains

Corollary

Let (W, S) be a simply-laced Coxeter system and let $\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_n}$ be a reduced expression for $w \in W$. Then $[\overline{w}]$ is a braid chain if and only if:

1. *n is odd and;*
2. $\mathcal{S}([\overline{w}]) = \{[1,3],[3,5],\ldots,[n-4,n-2],[n-2,n]\}$.

Example

The Coxeter system of type A_4 is determined by the following graph.

The word 1213243 is reduced. Let's apply all possible braid moves:

$$\underbrace{1213243}_{\overline{w}} = 2123243 = 2132343 = 2132434.$$

So $\mathcal{S}([\overline{w}]) = \{[1,3],[3,5],[5,7]\}$. By the above result, $[\overline{w}]$ is a braid chain.

Static support for braid shadows

Theorem (ABCE)

Assume that (W, S) *is a simply-laced Coxeter system such that the Coxeter graph has no three-cycles. If* $\overline{w}_1 = s_{x_1} s_{x_2} \cdots s_{x_n}$ *and* $\overline{w}_2 = s_{y_1} s_{y_2} \ldots s_{y_n}$ *are two braid equivalent reduced expressions for the same group element* $w \in W$*, then*

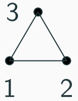
$$supp_{[i,i+2]}(\overline{w}_1) = supp_{[i,i+2]}(\overline{w}_2)$$

whenever $[i, i+2] \in \mathcal{S}(\overline{w}_1) \cap \mathcal{S}(\overline{w}_2)$*.*

Counter-example in $\tilde{A}_2$

Example

The simply-laced Coxeter system $\tilde{A}_2$ is determined by the following graph.



The expressions $\overline{w}_1 = 1213121$ and $\overline{w}_2 = 2123212$ are reduced. Moreover,

- $\overline{w}_1$ and $\overline{w}_2$ are braid equivalent and;
- $[3,5] \in \mathcal{S}(\overline{w}_1) \cap \mathcal{S}(\overline{w}_2)$.

Yet,

$$\text{supp}_{[3,5]}(\overline{w}_1) = \{1,3\} \neq \{2,3\} = \text{supp}_{[3,5]}(\overline{w}_2).$$

This shows that the previous result is false when the Coxeter graph has three-cycles.

Static support for $[\overline{w}]$

Theorem (ABCE)
Let (W, S) be a simply-laced Coxeter system whose Coxeter graph has no three-cycles. Let $\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_n}$ be a reduced expression for $w \in W$. If $[i, i+2] \in \mathcal{S}(\overline{w})$ such that $supp_{[i,i+2]}(\overline{w}) = \{s, t\}$, then $supp_{i+1}([\overline{w}]) = \{s, t\}$.

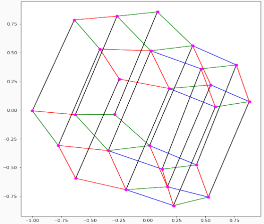
Example
The word $\overline{w} = 3431323$ is a reduced expression for some $w \in W(D_4)$. By applying all possible braid moves, we have $[\overline{w}] = \{3431323, 4341323, 3431232, 4341232, 3413123\}$. Note that $[1, 3] \in \mathcal{S}(\overline{w})$.

$$\text{supp}_{[1,3]}(\overline{w}) = \{3, 4\} \quad \Longrightarrow \quad \text{supp}_2([\overline{w}]) = \{3, 4\}$$

Future work

Future Work

- Generalize theory and results to arbitrary Coxeter systems.
- Investigate similar results in Coxeter systems whose Coxeter graphs have three-cycles.
- Find an elegant characterization of braid chains and links.
- Use current results to describe the structure of braid graphs in simply-laced Coxeter systems.



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The End.