Braid shadows in simply-laced Coxeter systems

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Simply-laced Coxeter systems

Definition

A simply-laced Coxeter system is a pair (W, S) consisting of a finite set S of generators and a group W, called a Coxeter group, given by the presentation

$$W = \langle S \mid s^2 = e,\ (st)^{m(s,t)} = e \rangle,$$

where $m : S \times S \to \{1, 2, 3\}$.

Remark

In a simply-laced Coxeter system, we have the following relations

$$m(s,t) = 2 \implies (st)^2 = e \implies st = ts \quad \Big\} \quad \textit{commutation relation}$$

$$m(s,t) = 3 \implies (st)^3 = e \implies sts = tst \quad \Big\} \quad \textit{braid relation}$$

Coxeter graphs

Definition

Let (W, S) be a simply-laced Coxeter system. A Coxeter graph has

1. vertex set S, and;
2. edges $\{s, t\}$ if and only if $sts = tst$.

We say that (W, S) is triangle-free if the corresponding Coxeter graph has no three-cycles.

Reduced Expressions

Definition

Let (W, S) be a Coxeter system and $w \in W$. A word $s_{x_1} s_{x_2} \cdots s_{x_n} \in S^*$ is called an expression for w if it is equal to w when considered as a group element of W. If n is minimal among all expressions for w, then the expression is reduced.

Matsumoto's Theorem for simply-laced Coxeter systems

Theorem (Matsumoto)

Let (W, S) be a simply-laced Coxeter system. Any two reduced expressions for the same group element $w \in W$ are related by a sequence of commutation ($st \mapsto ts$) and braid moves ($sts \mapsto tst$).

Braid equivalence

Definition

Let α and β be two reduced expressions for the same group element. We say that α and β are braid equivalent if we can obtain α from β via a sequence of braid moves.

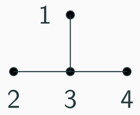
Remark

Braid equivalence is an equivalence relation on the set of reduced expressions for a fixed group element. The corresponding equivalence classes are called braid classes and are denoted by $[\alpha]$.

Braid moves

Example

The simply-laced Coxeter system of type D_4 is determined by the following graph.



The word $\zeta = 3134323$ is a reduced expression. There are many possible braid moves:

$$3134323 \mapsto 3143423$$

$$3134323 \mapsto 1314323$$

$$3134323 \mapsto 3134232$$

Applying all possible braid moves yields the braid class:

$$[\zeta] = \{1314232, 3134232, 3134323, 1314323, 3143423\}$$

Braid shadows

Definition

Let $\alpha = s_{x_1} s_{x_2} \cdots s_{x_n}$ be a reduced expression for $w \in W$. Write $[i, i+2] := \{i, i+1, i+2\}$.

1. The set of braid shadows for α is the set

$$\mathcal{S}(\alpha) := \{[i, i+2] : s_{x_i} = s_{x_{i+2}} \text{ and } m(s_{x_i}, s_{x_{i+1}}) = 3\}$$

2. The set of braid shadows for the entire braid class $[\alpha]$ is the set

$$\mathcal{S}([\alpha]) := \bigcup_{\beta \in [\alpha]} \mathcal{S}(\beta).$$

E.g. $\zeta = \underline{31\overline{3}}\overline{4}\underline{\overline{3}23}$ we have $\mathcal{S}(\zeta) = \{[1,3], [3,5], [5,7]\}$.

Braid shadows & links

Theorem (ABCE)

Assume that (W, S) is a simply-laced Coxeter system. Let $\alpha = s_{x_1} \dots s_{x_n}$ be a reduced expression for some $w \in W$. If $[i, i+2] \in \mathcal{S}([\alpha])$, then $[i+1, i+3] \notin \mathcal{S}([\alpha])$.

Remark

If you followed Dana's talk: we can now say $\alpha = s_{x_1} \dots s_{x_n}$ is a link if and only if:

- *n is odd;*
- $\mathcal{S}([\alpha]) = \{[1,3],[3,5],\dots,[n-4,n-2],[n-2,n]\}$.

We know that every reduced expression factors uniquely into a product of links.

Support

Definition

Let $\alpha = s_{x_1} s_{x_2} \cdots s_{x_n}$ be a reduced expression for $w \in W$. We define the support on an interval $[i, j] \subset \mathbb{N}$ as follows.

1. The support for α is the set $\operatorname{supp}_{[i,j]}(\alpha) := \{s_{x_k} : i \leq k \leq j\}$.
2. The support for $[\alpha]$ is the set

$$\operatorname{supp}_{[i,j]}([\alpha]) := \bigcup_{\beta \in [\alpha]} \operatorname{supp}_{[i,j]}(\beta).$$

Remark

If $i = j$, then we write i for $[i, i]$.

Support for braid shadows

Theorem (ABCE)

Assume that (W, S) is a simply-laced triangle-free Coxeter system. Consider the set

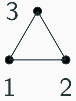
$$R(i) = \{\boldsymbol{\beta} \in [\boldsymbol{\alpha}] : [i, i+2] \in \mathcal{S}(\boldsymbol{\beta})\}.$$

Then $supp_{[i,i+2]}(\boldsymbol{\beta}) = supp_{[i,i+2]}(\boldsymbol{\gamma})$ for all $\beta, \gamma \in R(i)$.

Why do we need triangle-free?

Example

The simply-laced Coxeter system $\tilde{A}_2$ is determined by the following graph.



The reduced expressions $\boldsymbol{\alpha} = 12\underline{131}21$ and $\boldsymbol{\beta} = 21\underline{232}12$ are braid equivalent:

$$\boldsymbol{\alpha} = \underline{121}3121 \mapsto \underline{212}3\overline{121} \mapsto 2123\overline{212} = \boldsymbol{\beta}$$

Notice $[3,5] \in \mathcal{S}(\boldsymbol{\alpha}) \cap \mathcal{S}(\boldsymbol{\beta})$ while

$$\text{supp}_{[3,5]}(\boldsymbol{\alpha}) = \{1, 3\} \neq \{2, 3\} = \text{supp}_{[3,5]}(\boldsymbol{\beta}).$$

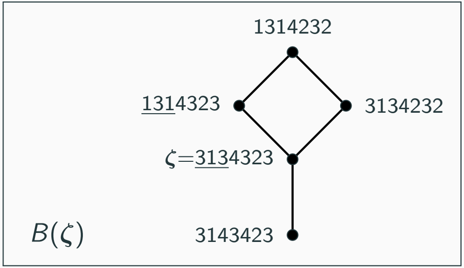
This shows that the previous result is false when the Coxeter graph has three-cycles.

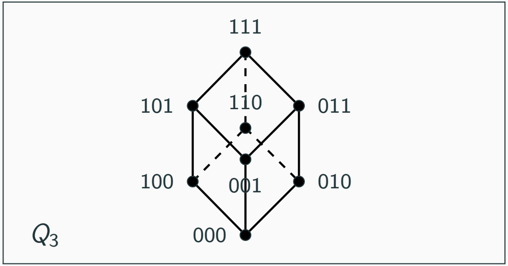
Invariant support across the braid class

Theorem (ABCE)

Let (W, S) be a simply-laced triangle-free Coxeter system. Let $\alpha = s_{x_1} s_{x_2} \cdots s_{x_n}$ be a reduced expression for $w \in W$. If $[i, i+2] \in \mathcal{S}(\alpha)$ such that $supp_{[i,i+2]}(\alpha) = \{s, t\}$, then $supp_{i+1}([\alpha]) = \{s, t\}$.

Application of results: braid graphs





Braid graph $B(\alpha)$

1. Vertices : $[\alpha]$
2. Edges: single braid moves

m-dimensional hypercube graph Q_m

1. Vertices: $\{0,1\}^m$ set of m-bit binary strings
2. Edges: differ by single digit

Application of results: braid graphs (cont.)

Definition

Let $\alpha = s_{x_1} s_{x_2} \cdots s_{x_n}$. Assume n is odd & $\mathcal{S}([\alpha]) = \{[1,3],[3,5],\ldots,[n-4,n-2],[n-2,n]\}$, i.e., α is a link. Define $m := |\,\mathcal{S}([\alpha])|$.

Assume that $\text{supp}_{2k}([\alpha]) = \{s_{2k}, t_{2k}\}$ where $m(s_{2k}, t_{2k}) = 3$ for each $k = 1, 2, \ldots, m$.

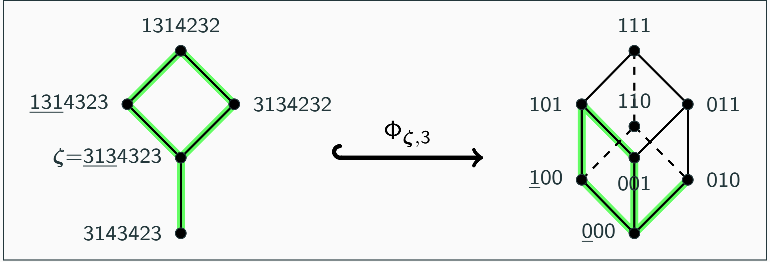
Define a map $\Phi_{\alpha,m} : [\alpha] \to \{0,1\}^m, \quad \beta \mapsto d_1 d_2 \cdots d_m$ where

$$d_k = \begin{cases} 0, & \text{if } \text{supp}_{2k}(\beta) = \{s_{2k}\} \\ 1, & \text{if } \text{supp}_{2k}(\beta) = \{t_{2k}\}, \end{cases} \quad \text{for } k = 1, 2, \ldots, m.$$

Embedding of braid graph for ζ into Q_3

Theorem (ABCE)
The map $\Phi_{\alpha,m}$ is a graph embedding of $B(\alpha)$ into Q_m, i.e., an edge-preserving injection.

Example
The reduced expression $\zeta = \underline{313}\overline{34}\underline{323}$ is a link: $\mathcal{S}([\zeta]) = \{[1,3],[3,5],[5,7]\}$.



The End.